

Applications of N-structures to Fuzzy Graph Theory

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ABSTRACT

Fuzzy graphs allow us to model uncertainty and ambiguity near a many of complex networks. In this article, N-fuzzy graphs are analysed, focusing on creating generalized models. To achieve this, we initiate our work by defining N-fuzzy sets before defining N-fuzzy relations to the fundamental structure. The N-fuzzy graphs have N-fuzzy bridges, N-fuzzy cut vertices, and N-fuzzy trees. All structural concepts mentioned were thoroughly assessed. All characterizations and important properties were determined and described, including N-fuzzy bridges, N-fuzzy cut vertices, and N-fuzzy trees. These finding is migrate many classical concepts, which have already been gained from the fields of crisp graphs and fuzzy graphs, to the N-fuzzy encompassing. The findings were clarified by working on specific and generalized examples. The findings gained from this N-fuzzy graph theories can be applied to aiding complex decisions and networks, especially those containing uncertainty and of varying degree systems.

Keywords

N-fuzzy graphs;
N-fuzzy trees;
N-fuzzy relation

1. Introduction

In 1965, L. A. Zadeh [1] introduced the notion of fuzzy subset of a set as a method of presenting uncertainty. The fuzzy systems have been used with success in last year's, in problems that involve the approximate reasoning. It has become a vast research area in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, computer networking, etc. [2] It is well known that graphs are simply models of relations. A graph is an easy way of representing information involving relationship between objects. The objects are represented by vertices and relation by edges. M. Akram [3] applied N-structures to graph theory and he defined N-graphs

A graph is a pair (U, R) , where U is a set and R is a relation on U [4, 5]. The elements of U are thought of as vertices of the graph and the elements of R thought of as the edges. Similarly, any N-fuzzy relation α on a N-fuzzy subset f of a set U can be regarded as defining a weighted graphs, or N-fuzzy graphs, where the edge $(u, v) \in U \times U$ has strength $\alpha(u, v) \in [-1, 0]$. The notation (u, v) is used for an edge, where $u, v \in U$. An N-fuzzy graph $G = (U, f, \alpha)$ is a non empty set U together with a pair of functions

$$f: U \rightarrow [-1, 0] \text{ and } \alpha: U \times U \rightarrow [-1, 0] \text{ such that for } u, v \in U, \\ \alpha(u, v) \geq f(u) \vee f(v).$$

We call f an N-fuzzy vertex set of G and α is an N-fuzzy edge set of G , respectively. Note that α is an N-fuzzy relation on f .

A sufficient condition for node to be an N-fuzzy cut node is obtained which becomes also necessary in the case of N-fuzzy trees. A characterization of N-fuzzy cut nodes is obtained for N-fuzzy graphs G such that G^* is a cycle [6-9]. Some significant differences from the crisp theory are pointed out. Note that, bridges and cathodes of the crisp graph G^* are N-fuzzy bridge and N-fuzzy cut nodes of the N-fuzzy graph G respectively. [10] Next we present a necessary condition for an arc (u, v) to be an N-fuzzy bridge and prove that this condition is also sufficient in the case of N-fuzzy trees. Also N-fuzzy bridges and N-fuzzy cut nodes are characterized using maximum spanning tress. The concept of maximum spanning tree plays a key role in the characterization of N-fuzzy trees. An N-fuzzy graph is an

N-fuzzy tree if and only if it has unique maximum spanning tree [11 12]. A sufficient condition for an N-fuzzy graph to be an N-fuzzy tree is also obtained by using the concept of N-fuzzy cycle.

2. Preliminaries

Definition 2.1: An N-fuzzy set f (N-fuzzy sub set of non-empty set S) is defined as mapping

$$f: S \rightarrow [-1,0],$$

Where $f(x)$ is the negative membership degree of x to the N-fuzzy set S . We denote by $NF(S)$ the collection of all N-fuzzy subsets of S .

Definition 2.2: [3] Let S and T be two non-empty sets and let f and g be N-fuzzy subsets of S and T , respectively. Then an N-fuzzy relation α of $S \times T$ is a relation such that $\alpha(x, y) \geq f(x) \vee g(y), \forall x \in S$ and $y \in T$.

Definition 2.3: [3] Let S be a non-empty set. An N-fuzzy graph is a pair of functions. $G: (f, \alpha)$, where f is an N-fuzzy subset of non-empty set S and α is symmetric N-fuzzy relation on f .

i.e.

$$f: S \rightarrow [-1,0] \text{ and}$$

$$\alpha: S \times S \rightarrow [-1,0] \text{ such that } \alpha(\mu, \nu) \geq f(\mu) \vee f(\nu) \quad \forall \mu, \nu \in S.$$

Definition 2.4: An arc (μ, ν) is an N-fuzzy bridge of $G: (f, \alpha)$ if the deletion of (μ, ν) reduces the strength of connectedness between same pair of nodes. Equivalently, (μ, ν) is an N-fuzzy bridge if and only if there are nodes x, y such that (μ, ν) is an arc of every strongest x - y path.

Definition 2.5: A node is an N-fuzzy cut node of $G: (f, \alpha)$ if removal of it reduces the strength of connectedness between same other pair of nodes. Equivalently, w is an N-fuzzy cut node if and only if there are u and v distinct from w such that w is on every strongest u - v path.

Example 2.1: In figure 2.1 $(u_1, u_3), (u_3, u_2), (u_2, u_4), (u_4, u_5)$ are the N-fuzzy bridges and u_2, u_3, u_4 are the N-fuzzy cut nodes of an N-fuzzy graph $G: (f, \alpha)$.

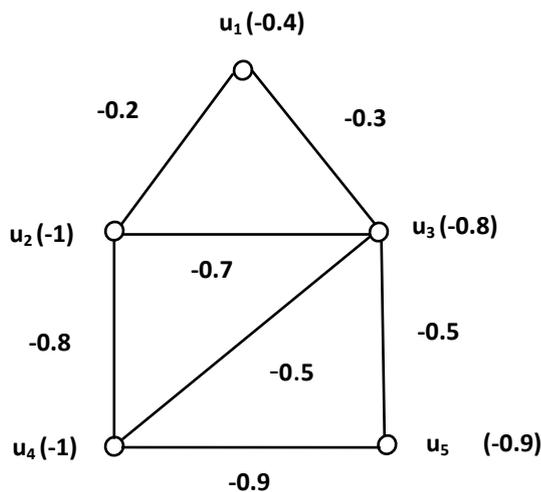


Fig.2.1: An N-fuzzy graph $G: (f, \alpha)$

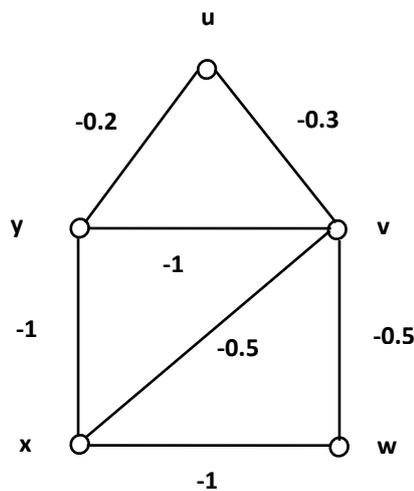


Fig.2.2 (a) An N-fuzzy tree $G: (f, \alpha)$

Definition 2.6: A connected N-fuzzy graph $G: (f, \alpha)$ is an N-fuzzy tree if it has an N-fuzzy spanning sub graph $F: (f, \beta)$, which is tree, where for all arcs (u, v) not in F $\alpha(u, v) > \alpha^\infty(u, v)$. Equivalently, there is a path in F between u and v whose strength exceeds $\alpha(u, v)$ for all (u, v) not in F . If G is such that G^* is a tree, then F is G itself.

Example 2.2: The Fig. 2.2(a) is an N-fuzzy tree $G: (f, \alpha)$ and Fig 2.2(b) is an Spanning sub graph $F: (f, \beta)$.

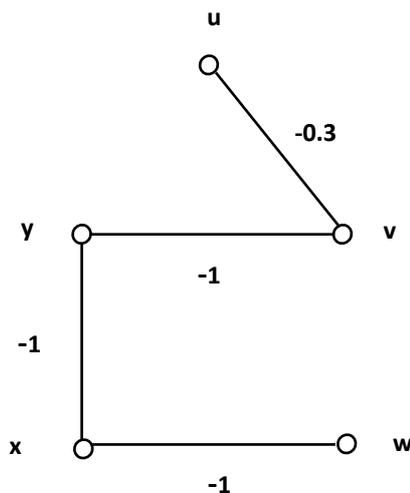


Fig.2.2 (b) Spanning sub graph $F: (f, \beta)$

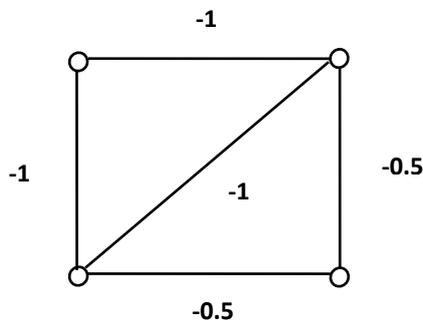
Here $\alpha(u, y) = -0.2 > -0.3 = \beta^\infty(u, y)$, $\alpha(v, w) = -0.5 > -1 = \beta^\infty(v, w)$ and $\alpha(v, x) = -0.5 > -1 = \beta^\infty(v, x)$.

Definition 2.7: Let $G: (f, \alpha)$ be an N-fuzzy graph such that G^* is cycle. Then, G is called an N-fuzzy cycle if it has more than one weakest arc.

Definition 2.8: A connected N-fuzzy graph $G: (f, \alpha)$ with no N-fuzzy cut nodes is called a block.

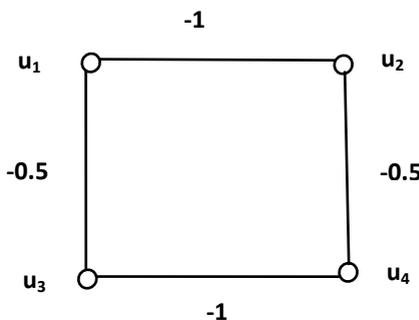
In [13, 14] it was noted that blocks in N-fuzzy graphs may have N-fuzzy bridges. In next example G_1 is a block without N-fuzzy bridge, and G_2 is a block with N-fuzzy bridges (u_1, u_2) and (u_3, u_4) .

Example 2.3: Let G_1 be a block without N-fuzzy bridge and G_2 a block with N-fuzzy bridges (u_1, u_2) and (u_3, u_4) See Figs 2.3(a) & 2.3 (b).



G_1

Fig.2.3 (a)



G_2

Fig.2.3 (b)

Theorem 2.1: An arc (a node) is an N-fuzzy bridge (N-fuzzy cut node), if and only if there exists a partition V of nodes into subsets U, W Such that all nodes $u \in U$ and $w \in W$, the arc (the node) is on every strongest u - v path.

Theorem 2.2: Let $G: (f, \alpha)$ be an N -fuzzy graph with $U = u_1, u_2, \dots, u_n$. And let C be the cycle $u_1, u_2, \dots, u_n, u_1$. If $C \subseteq \alpha^*$ and for every arc $(u_j, u_k) \in \alpha^* - C$, $\alpha(u_j, u_k) < \text{Max}\{\alpha(u_i, u_{i+1}) : i = 1, 2, 3, 4, \dots, n\}$ where, $u_{n+1} = u_1$ then either α is a constant on C or G has an N -fuzzy bridge.

Theorem 2.3: Let G be a connected N -fuzzy graph. If there is one strongest path between any two nodes of G , then G is called an N -fuzzy tree.

Theorem 2.4: If G is an N -fuzzy tree, then arcs of α are the N -fuzzy bridges of G .

Theorem 2.5: The following statements are equivalent for an arc (u, v) of an N -fuzzy graph $G: (f, \alpha)$.

- i. (u, v) is an N -fuzzy bridge.
- ii. (u, v) is not a weakest arc of any cycle in G .

Theorem 2.6: An arc (a node) is an N -fuzzy bridge (N -fuzzy cut node), if and only if there exists a partition V of nodes into subsets U, W Such that all nodes $u \in U$ and $w \in W$, the arc (the node) is on every strongest u - v path.

Theorem 2.7: Let $G: (f, \alpha)$ be an N -fuzzy graph with $U = u_1, u_2, \dots, u_n$. And let C be the cycle $u_1, u_2, \dots, u_n, u_1$. If $C \subseteq \alpha^*$ and for every arc $(u_j, u_k) \in \alpha^* - C$, $\alpha(u_j, u_k) < \text{Max}\{\alpha(u_i, u_{i+1}) : i = 1, 2, 3, 4, \dots, n\}$ where, $u_{n+1} = u_1$ then either α is a constant on C or G has an N -fuzzy bridge.

Theorem 2.8: Let G be a connected N -fuzzy graph. If there is one strongest path between any two nodes of G , then G is called an N -fuzzy tree.

Theorem 2.9: If G is an N -fuzzy tree, then arcs of α are the N -fuzzy bridges of G .

Theorem 2.10: Let $G: (f, \alpha)$ be an N -fuzzy graph and (u, v) be an N -fuzzy bridge of G . Then, $\alpha^\alpha(u, v) = \alpha(u, v)$.

Proof: Suppose that (u, v) is an N -fuzzy bridge and that $\alpha^\alpha(u, v)$ exceeds $\alpha(u, v)$. Then there exists a strongest u - v Path with strength greater than $\alpha(u, v)$ and all arcs of this strongest path have strength greater than $\alpha(u, v)$. Now, this path to gather with the arc (u, v)

Remark 2.1: It follows that an arc (u, v) is an N -fuzzy bridge if and only if it is the unique strongest u - v path. However, the converse of this theorem is not true.

In the following N -fuzzy graph (Fig 2.4), (u_2, u_4) , and (u_3, u_4) are the only N -fuzzy bridges and $\alpha^\alpha(u_1, u_2) = \alpha(u_1, u_2) = 0.4 = \alpha^\alpha(u_1, u_3) = \alpha(u_1, u_3)$ but (u_1, u_2) and (u_1, u_3) are not N -fuzzy bridges. The condition for the converse to be true is discussed next.

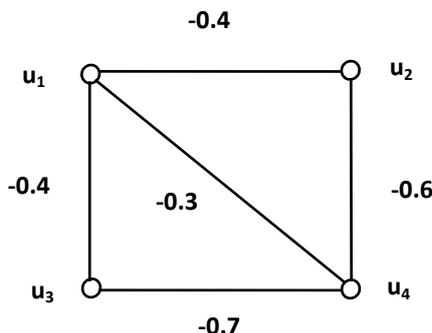


Fig.2.4

We first observe that the identification of N -fuzzy cut nodes is not easy. In the next Theorem, we characterize N -fuzzy cut nodes in G such that G^* is a cycle and present a sufficient condition for a node to be a fuzzy cut node in the general case.

Theorem 2.11: Let $G: (f, \alpha)$ be an N -fuzzy graph such that G^* is a cycle. Then, a node is an N -fuzzy cut node of G if and only if it is a common node of two N -fuzzy bridges.

Proof: Let w be an N -fuzzy cut node of G . Then, there exists u and v , distinct from w , such that w is on every strongest u - v path which is unique since G^* is a cycle and it follows that all its arcs are N -fuzzy bridges. Thus w is a common node of two N -fuzzy bridges.

Conversely, let w be a common node of two N -fuzzy bridges (u, w) and (w, v) . Then both (u, w) and (u, v) are not the weakest arcs of G . Also, the path from u to v not containing the arcs (u, w) and (w, v) has strength less than $\alpha(u, w) \vee \alpha(w, v)$. Thus the strongest u - v path is the path u, w, v and $\alpha^\alpha(u, v) = \alpha(u, w) \vee \alpha(w, v)$. Thus w is an N -fuzzy cut node. In general we have,

Theorem 2.12 Let $G: (f, \alpha)$ be an N -fuzzy graph, and let w be a common node of two N -fuzzy bridges. Then w is an N -fuzzy cut node.

Proof: Let (u_1, w) and (w, u_2) be two N -fuzzy bridges. Then there exists some u, v such that (u_1, w) is on every strongest u - v path. If w is distinct from u and v it follows that w is an N -fuzzy cut node. Next, suppose one of v, u is w so that (u_1, w) is on every strongest u - w path or (w, u_2) is on every strongest w - v path. If possible let w be not an N -fuzzy cut node. Then between every two nodes, distinct from w , there exists at least one strongest path not

containing w . In particular there exists at least one strongest path P , joining u_1 , and u_2 , not containing w . This path together with (u_1, w) and (w, u_2) forms a cycle. There are two cases.

Case: 1: Let u_1, w, u_2 is not a strongest path. Then, either (u_1, w) or (w, u_2) or both become the weakest arcs of the cycle which contradicts that (u_1, w) and (w, u_2) are N-fuzzy bridges.

Case: 2: Let u_1, w, u_2 is a strongest path joining u_1 to u_2 . Then, $\alpha^\times(u_1, u_2) = \alpha(u_1, w) \vee \alpha(w, u_2)$ the strength of P . Thus, arcs of P are as strong as $\alpha(u_1, w)$ and $\alpha(w, u_2)$ which implies that (u_1, w) , (w, u_2) or both are the weakest arcs of the cycle, which again is a contradiction.

Remark 2.2: The condition in the above theorem is not necessary. In Fig. 2.5, w is the N-fuzzy cut node; (u, w) and (v, x) are the only N-fuzzy bridges, in Fig. 2.6, w is the fuzzy cut node (u, w) is the only N-fuzzy bridge and (u, w) is the only N-fuzzy bridge and in Fig. 2.7, w is the N-fuzzy cut node and no arc is a fuzzy bridge. But the converse of theorem 2.12 holds in N-fuzzy trees.

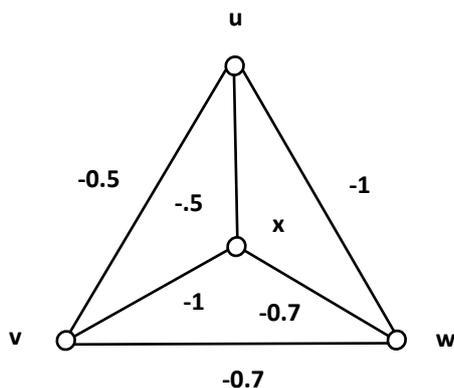


Fig.2.5

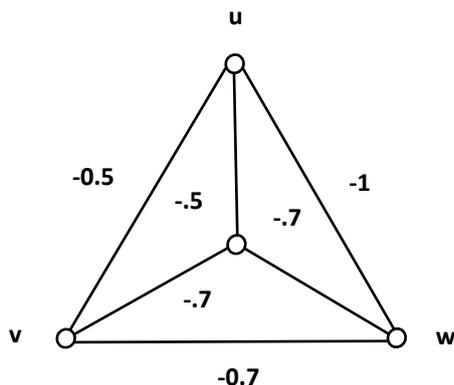


Fig.2.6

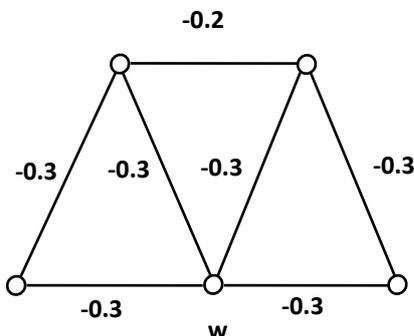
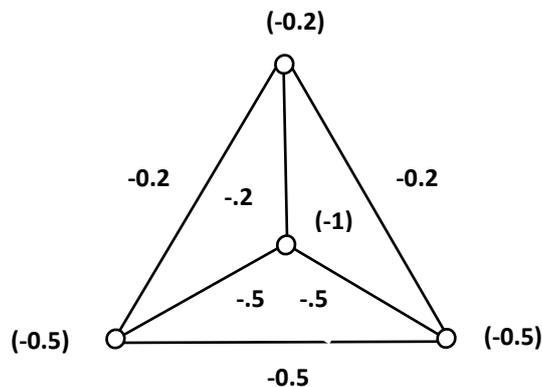


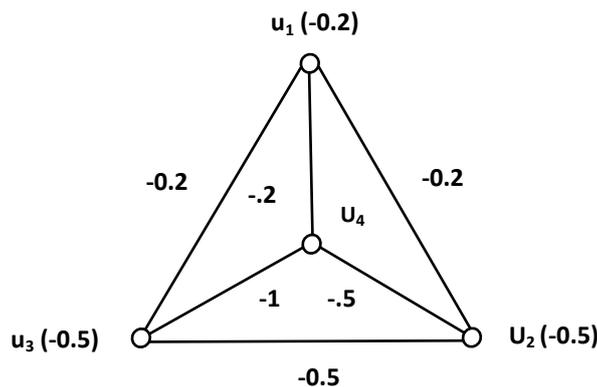
Fig.2.7

Remark 2.3: As distinct from crisp graph theory, there are N-fuzzy graphs with N-fuzzy bridge and having no N-fuzzy cut node. Thus (u, w) and (v, x) are the N-fuzzy bridges and no node is an N-fuzzy cut node, where $0 > a > b \geq -1$.

Lemma 2.1: If $G: (f, \alpha)$ is a complete N-fuzzy graph, then $\alpha^\times(u, v) = \alpha(u, v)$.



G_1
Fig.2.9 (a)



G_2
Fig.2.9 (b)

Now using the concept of maximum spanning tree of an N-fuzzy graph, we present a characterization of N-fuzzy bridge and N-fuzzy cut node. Also, in a (crisp) graph G^* , note that each spanning tree is a maximum spanning tree. The following are characterization of N-fuzzy bridge and N-fuzzy cut node, which are obvious in crisp case.

Theorem 2.15: An arc (u, v) is an N-fuzzy bridge of $G: (f, \alpha)$ if and only if (u, v) is in every maximum spanning tree of G .

Proof: Let (u, v) be an N-fuzzy bridge of G , then arc (u, v) is the unique strongest $u-v$ path and hence is in every maximum spanning tree of G .

Conversely, let (u, v) be in every maximum spanning tree T of G and assume that (u, v) is not an N-fuzzy bridge. Then (u, v) is a weakest arc of some cycle in G and $\alpha^\infty(u, v) < \alpha(u, v)$, which implies that (u, v) is in no maximum spanning tree of G .

Remark 2.5: From Theorem 2.15, it follows that arcs not in T are not N-fuzzy bridges of G and we have the following corollary.

Corollary: If $G: (f, \alpha)$ is a connected N-fuzzy graph with $|U| = n$ then G has at most $n-1$ N-fuzzy bridge.

Theorem 2.16: A node w is an N-fuzzy cut node of $G: (f, \alpha)$ if and only if w is an internal node of every maximum spanning tree of G .

Proof: Let w be an N-fuzzy cut node of G . Then there exist u, v distinct from w such that w is on every strongest $u-v$ path. Now each maximum spanning tree of G contains unique strongest $u-v$ path and hence w is an internal node of each maximum spanning tree of G .

Conversely, let w be an internal node of every maximum spanning tree. Let T be a maximum spanning tree and let (u, w) and (w, v) be arcs in T . Note that the path u, w, v is a strongest $u-v$ path in T . If possible assume that w is not an N-fuzzy cut node. Then between every pair of nodes u, v there exist at least one strongest $u-v$ path not containing w . Consider one such $u-v$ path P which clearly contain arcs not in T . Now without loss of generality, let $\alpha^\infty(u, v) = \alpha(u, w)$ in T . Then arcs in P have strength $\leq \alpha(u, w)$. Removal of (u, w) and adding P in T will result in another maximum spanning tree of G for which w is an end node, which contradicts our assumption.

Remark 2.6: It follows from Theorem 2.16, that the end nodes of maximum spanning tree T of G are not N-fuzzy cut nodes of G . This results in the following corollary.

Corollary: [15] Every N-fuzzy graph has at least two nodes which are not N-fuzzy cut nodes of G. However, we see that there are N-fuzzy graphs with diametrical nodes, nodes which have maximum eccentricity, as N-fuzzy cut nodes, distinct from crisp graph theory.

3. N-Fuzzy Trees

A tree is a connected graph that contain no cycle. In a tree, every pair of points is connected by a unique path. Each vertex has at most one incoming edge. Rosenfeld has proved that if there exists a unique strongest path joining any two nodes in G then G is a fuzzy tree and the converse does not hold, similarly in N-fuzzy sets, if there exists a unique strongest path joining any two nodes in G then G is an N-fuzzy tree and the converse does not hold. In Fig. 3.1, G is an N-fuzzy tree and $P_1: x, u, v, w, y$ & $P_2: x, v, w, y$ are two strongest x - y paths with $\alpha^\alpha(u, v) = -0.5$ of which P_2 is in F. Also note that if G^* is a tree, then F is G itself and maximum spanning tree T of an N-fuzzy tree G is the required N-fuzzy spanning sub graph and that T is unique for an N-fuzzy tree.

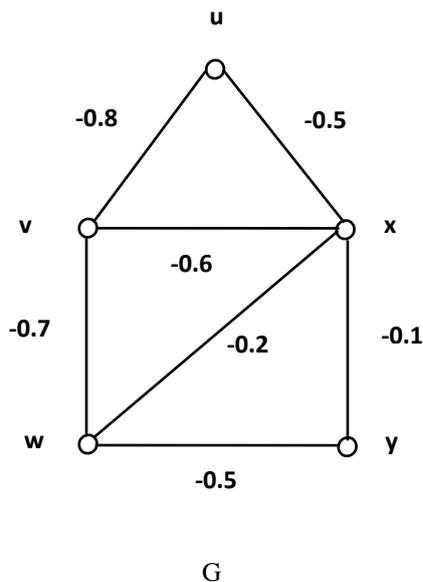


Fig.3.1 (a)

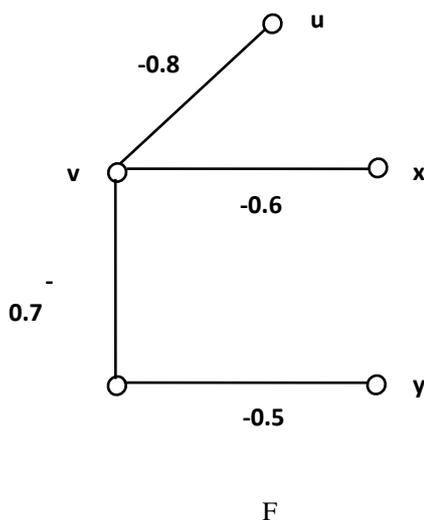


Fig.3.1 (b)

Lemma 3.1 If (g, β) is a partial N-fuzzy sub graph of (f, α) , then for all u, v $\beta^\alpha(u, v) \geq \alpha^\alpha(u, v)$.

Theorem 3.1 If $G: (f, \alpha)$ is an N-fuzzy tree and $G^* (f^*, \alpha^*)$ is not a tree, then there exists atleast one arc (u, v) in α^* for which $\alpha^\infty(u, v) < \alpha(u, v)$.

Proof: If G is an N -fuzzy tree, then by definition there exists an N -fuzzy spanning sub graph $F:(f, \alpha)$ which is a tree and $\alpha^\infty(u, v) < \alpha(u, v)$ for all arcs (u, v) not in F . Also $\beta^\alpha(u, v) \geq \alpha^\alpha(u, v)$ by lemma 3.3. Thus $\alpha^\infty(u, v) < \alpha(u, v)$ for all (u, v) not in F , and by hypothesis there exists atleast one arc (u, v) not in F , which completes the proof.

Theorem 3.2 Let G : be an N -fuzzy tree and $G^* \neq K_1$. Then G is not complete.

Proof: If possible let G be a complete N -fuzzy graph. Then $\alpha^\infty(u, v) = \alpha(u, v)$ for all u, v . [Lemma 3.1] Now G being an N -fuzzy tree, $\alpha^\infty(u, v) < \alpha(u, v)$ for all (u, v) not in F . Thus $\alpha^\alpha(u, v) > \beta^\alpha(u, v)$, contradicting lemma 3.3.

Remark 3.1 It is proved that if G is an N -fuzzy tree, then arcs of F are the N -fuzzy bridge of G and thus F is unique. In the next theorem we characterize N -fuzzy tree.

Theorem 3.4 If G is an N -fuzzy tree, then internal nodes of F are the N -fuzzy cut nodes of G .

Proof: Let w be any node in G . which is not an end node of F . Then, it is the common node of at least two arcs in F which are the N -fuzzy bridges of G and by Theorem 3.3, w is an N -fuzzy cut node. Also, if w is end node of F , then w is not an N -fuzzy cut node; for if so, and there exist u, v distinct from w such that w is on every strongest u - v path and one such path certainly lies in F . But w being an end node of F , this is not possible. With reference to the Remark 3.1, we have,

Corollary: An N -fuzzy cut node of an N -fuzzy tree is the common node of at least two N -fuzzy bridges. We have seen that the condition that an arc (u, v) is a strongest u - v path is only necessary for it to be an N -fuzzy bridge [Theorem3.1]. In the case of N -fuzzy trees this condition becomes sufficient also and we have the following characterization of N -fuzzy trees.

Theorem 3.5 $G:(f, \alpha)$ is an N -fuzzy tree if and only if the following are equivalent.

- (1) Arc (u, v) is an N -fuzzy bridge, $\alpha^\alpha(u, v) = \alpha(u, v)$.

Proof: Let $G:(f, \alpha)$ be an N -fuzzy tree, and let (u, v) be an N -fuzzy bridge. Then $\alpha^\alpha(u, v) = \alpha(u, v)$ [Theorem3.1]. Now let (u, v) be an arc of G such that, $\alpha^\alpha(u, v) = \alpha(u, v)$. If G^* is a tree, then clearly (u, v) is an N -fuzzy bridge; otherwise, it follows from Theorem 3.8 that (u, v) is in F and it is an N -fuzzy bridge, [Theorem 3.4].

Conversely, assume that (1) \Leftrightarrow (2). Construct a maximum spanning tree $T:(f, \beta)$ for G . If (u, v) is in T , by an algorithm $\alpha^\alpha(u, v) = \alpha(u, v)$ and hence (u, v) is an N -fuzzy bridge. Now these are the only N -fuzzy bridges of G for, if possible let (u', v') be an N -fuzzy bridge of G which is not in T . Consider a cycle C consisting of (u', v') and the unique u' - v' path in T . Now, the arcs of this u' - v' path being N -fuzzy bridges they are not weakest arcs of C and hence (u', v') must be the weakest arc of C and hence cannot be an N -fuzzy bridge [theorem3.4].

Moreover, for all arcs (u', v') not in T , we have $\alpha(u', v') > \beta^\alpha(u', v')$ for, If possible let $\alpha(u', v') \leq \beta^\alpha(u', v')$. But $\alpha(u', v') > \beta^\alpha(u', v')$ (strict inequality holds since (u', v') is not an N -fuzzy bridge). So $\beta^\alpha(u', v') > \alpha^\alpha(u', v')$ which gives a contradiction, since $\beta^\alpha(u', v')$ is the strength of the unique u' - v' path in T and by an algorithm $\beta^\alpha(u', v') = \alpha^\alpha(u', v')$ Thus T is the required spanning sub graph F , which is a tree and hence G is an N -fuzzy tree.

Remark 3.2: It follows from the proof of Theorem 3.5 that arcs of the maximum spanning tree T are the N -fuzzy bridges of the N -fuzzy tree G and thus we have,

An N -fuzzy graph is an N -fuzzy tree if and only if it has a unique maximum spanning tree.

Note that G is an N -fuzzy graph on n nodes then the maximum number of N -fuzzy bridge in G is in $n-1$ [Corollary to theorem 3.4] and it follows from Remark 3.2 that an N -fuzzy tree on n nodes have $(n-1)$ N -fuzzy bridges. In the following theorem, we consider the case of a general N -fuzzy graph.

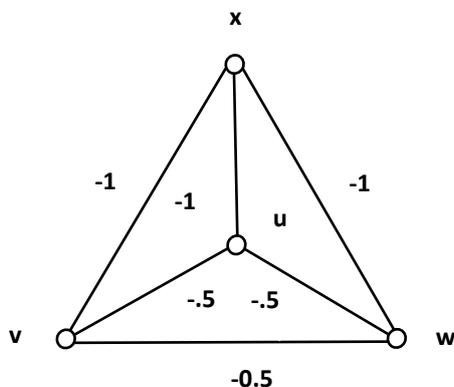


Fig.3.2

Theorem 3.6: Let $G:(f, \alpha)$ be a connected N -fuzzy graph with no N -fuzzy cycles. Then G is an N -fuzzy tree.

Proof: If G^* has no N-fuzzy cycles then G^* is a tree and G is an N-fuzzy tree. So assume that G has cycles and by hypothesis no cycle is an N-fuzzy cycle, *i.e.* every cycle in G will have exactly one weakest arc in it. Remove the weakest arc (say) e in a cycle C of G . If there are still cycles in the resulting N-fuzzy graph, repeat the process, which will eventually results is an N-fuzzy sub graph, which is a tree, and which is the required spanning sub graph F , hence proved.

Remark 3.3 Converse of the above theorem is not true. In Fig.3.2, G is an N-fuzzy tree and u, v, w, u is an N-fuzzy cycle.

4. Conclusion

In this article, we developed a rigorous theoretical framework for N-fuzzy graphs, building upon classical and fuzzy graphs and extending them to a more generalized N-fuzzy approach. N-fuzzy sets and N-fuzzy relations were employed to delineate N-fuzzy graphs, allowing the representation of complex networks characterized by multiple degrees of uncertainty. The primary structural elements, such as N-fuzzy bridges, N-fuzzy cut vertices, and N-fuzzy trees, were defined and thoroughly studied. Several of their characterizations and fundamental structural properties were described, enhancing the understanding of N-fuzzy graphs and their behavior with respect to connectivity. A number of examples showed the applicability and usefulness of the proposed results and the N-fuzzy graphs we defined. This study generalized results that can be found elsewhere in the fuzzy graph theory and opens several new possibilities for future work. Some of these relate to the study of N-fuzzy cycles and their connectivity, the study of domination and coloring problems, as well as to the use of N-fuzzy graphs in practical problems such as communication networks, decision-making systems, and uncertainty modeling. The work done in this paper constitutes a springboard for the further theoretical and practical development of the theory of N-fuzzy graphs and the applications that derive from it.

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