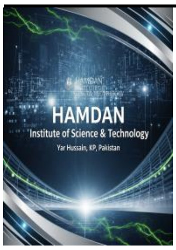

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Development of Intelligent Based Fuzzy Decision-Making Model Through ff'-Fractional Fuzzy Information

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ABSTRACT

Making reliable and precise assessments is challenging in the context of fuzzy sets (FSs) due to hesitancy and uncertainty in non-membership degree (NMD). Researchers have been developing methods to reduce these issues, which has encouraged the creation of novel decision support tools. Therefore, in this paper, we introduce a new structure called ff'-fractional fuzzy sets (ff'-FFSs). The proposed structure is the extension of p, q-rung orthopair fuzzy sets and fractional fuzzy sets, where and are different fractional parameters. This provides an extra space for the experts to convey their information more accurately. Following that, we define a series of arithmetic and weighted aggregation operators to aggregate expert information into combined assessment information. Next, we propose a new method called ff'-fractional fuzzy combinative distance-based assessment ff'-FF CODAS method. This method assesses alternatives using Euclidean and taxicab distances, which enhances its ability to distinguish strongly evaluated or opposing alternatives, resulting in more consistent, accurate rankings. With the addition of ff'-FF information, this method becomes more reliable, precise, and superior, enhancing assessment accuracy. After that, the proposed method is used for the selection of the best supervisor for higher studies, which is a complex decision-making problem. Thus for this, we collect the information from three experts, and hence the result is calculated. Furthermore, we conduct a comparative study to check the success and ability of the proposed method while comparing it with existing methods and operators. Lastly, a sensitivity analysis is conducted based on two parameters, f and f', to assess the stability of the proposed method. The results of the comparison and sensitivity demonstrate that the proposed method is effective for a wide range of decision-making problems.

Keywords

Supervisor selection; ff'-fractional fuzzy sets; Multi-criteria decision-making; CODAS method

1. Introduction

A crucial first stage in the student research career is choosing a Ph.D. (Doctor of Philosophy) supervisor. In fact, this has become arguably the most important factor influencing the successful completion of academic programs. Before selecting a Ph.D. adviser, it is crucial for everyone to verify each component of the supervisor's background to determine their usefulness. Seeking favorable responses to more than a few of these inquiries, including "Have individuals published studies just lately or not?" is advised (Mandal et al. [1]). Have they obtained any agreements or funds for research? Are individuals invited to attend seminars in overseas universities'' [2]. Additionally, students may be interested in the professors' publications, seminars, collaborations, past and present students, and other related information. As a result, pupils may take advantage of a framework that assists individuals in identifying essential requirements and directs their evaluation of supervisors based on those requirements. Thus, there are several approaches for choosing the most suitable Ph.D. supervisor. Hence, creating a novel approach for selecting the best Ph.D. supervisor is crucial.

1.1. A brief review of fuzzy sets and its extensions

The term "multi-criteria decision-making" (MCDM) [3-4] refers to the problems that occur when comparing several alternatives based on particular criteria. Decision-making may be classified as either single or group-oriented based on the number of experts. Multi-criteria group decision-making (MCGDM) [5-6] is a type of collective decision-making in which a group of individuals jointly assesses options to determine which is best. However, expert information can often be unclear, imprecise, or ambiguous, making it challenging to acquire accurate and trustworthy results. To overcome these difficulties, Zadeh [7] developed fuzzy sets (FS). These sets give for each element a membership degree (MD) inside the closed interval of [0, 1], providing a versatile and accurate way to express uncertainty that goes beyond conventional binary logic. However, FS is unable to accurately record circumstances in which non-membership facts are crucial. To get around this, Atanassov [8] created intuitionistic fuzzy sets (IFS), which allow for a more comprehensive modelling of uncertainty in decision-making by including both MD and NMD details. Even if IFS increases adaptability, it remains unable to accurately reflect the requirements of experts in particularly complicated situations. Therefore, Yager [9-10] addressed this limitation by developing Pythagorean fuzzy sets (PFS), which provide a more general and powerful structure for modeling uncertainty and are commonly used in decision-making applications. At that time, PFS was considered the most generalized structure of the existing ones. However, PFS is still limited in complex real-world scenarios, motivating the creation of q-rung orthopair fuzzy sets (q-ROFS) [11], which broaden prior structures to improve accuracy and flexibility by a parameter q. q-ROFS is a more generalized structure than the existing ones, because for $q = 1$, it is IFS, and for $q = 2$, it is PFS. The q-ROFS structure discusses the power q to be a positive integer greater than or equal to 1. As we know that there are infinitely many numbers between any two positive integers, q-ROFS was unable to discuss these values. To manage these drawbacks, Abdullah et al. [12] proposed fractional fuzzy sets (FFS), which extend q-ROFS by including a continuous fractional parameter f, which is in the form of p/q greater than or equal to 1. Thus, FFS results in an amazingly adaptive and robust structure for modelling uncertainty in uncertain and imprecise information. FFS has been developed in subsequent investigations for practical applicability in a variety of decision-making scenarios. It has been utilized to assess emotional intelligence [13], water filtration decision-making [14], solving three-way decision-making issues [15], and industrial robot selection [16]. These studies show that FFS is extremely successful in dealing with ambiguity, uncertainty, and ambiguous information in complicated decision-making circumstances. However, in the structure of FFS, the MD and NMD both have the same power f. This means that no work has been done on different powers of MD and NMD. For this reason, in this work, we established a novel notion of fractional fuzzy set (ff'-FFS), where the MD has the fractional parameter f, and the NMD has a fractional parameter f' such that both are in the form of p/q and not equal. Thus, the proposed structure is a more generalized structure than all the existing ones.

1.2. A brief overview of multi-criteria decision-making methods

Multi-criteria decision-making (MCDM) techniques offer a systematic framework for evaluating and comparing alternatives based on various, sometimes competing criteria. These techniques help experts identify the best optimal alternative by carefully addressing alternatives across criteria. Numerous approaches have been suggested and developed in order to deal with MCDM, among which is the Combinative Distance-Based Assessment (CODAS) approach. The CODAS method has been considered one of the numerous successful techniques suggested by Keshavarz Ghorabaee [17]. This approach combines two distance measures (Euclidean and taxicab) to provide reliable evaluation findings for alternatives. However, this approach was ineffective in a fuzzy environment. To address this limitation, Keshavarz Ghorabaee et al. [18] suggested a fuzzy CODAS framework that uses fuzzy weighted Hamming distance (HD) along with fuzzy weighted Euclidean distance (ED) instead of classical distances. The CODAS approach successfully handles fuzzy decision-making situations. Many researchers have looked into the CODAS approach after its first proposal. Pamucar et al. [19] proposed a linguistic neutrosophic CODAS model, while Bolturk and Kahraman [20] developed a novel CODAS model using interval-valued intuitionistic fuzzy information. In addition, several other MCDM methods have been introduced in recent years. Several significance models, including the weighted sum model (WSM) [21], weighted product model (WPM) [22], weighted aggregated sum product assessment (WASPAS) [23], analytical hierarchy process (AHP) [24], TODIM [25], TOPSIS approach [26], grey relational analysis (GRA) [27], and evaluation using distance from average solution EDAS [28], as well as multi-MOORA (multi-objective optimization ratio analysis) [29], are significant approaches. WSM is definitely one of the most often employed methods. This approach determines the best alternative using the 'additive benefit' hypothesis. WPM is very comparable to WSM. This methodology involves multiplying powered weighted rates (performance) rather than summing weighted rates, as in WSM. The WASPAS approach combines WSM and WPM approaches, offering benefits from each. The TOPSIS approach [30] is a value-based compensating approach. This method ranks alternatives according to their distance from the ideal as well as the nadir (positive as well as negative ideal solutions). The COPRAS approach [31] is a successful MCDM approach that prioritizes the most effective alternative depending on a ratio of benefit along with cost criteria achievement summing.

Therefore MCDM methods [32-33] play a vital role in decision-making [38] for selecting the best alternative, based on conflicting criteria.

1.3. Motivation of the work

The research mentioned above indicates that a number of ideas have recently been developed that are helpful in resolving DM issues. To the best of our expertise, there is no concept or implementation of fractional fuzzy sets that contains different fractional values for the power of MD and NMD for assessing decision-making issues. The present research is therefore inspired by the need to develop an updated tool that can deal with unpredictability in decision-making challenges: the ff'-fractional fuzzy set (ff'-FFSs). This is achieved by expanding the fractional fuzzy set to provide a broader and more reliable decision environment through the distinct fractional powers of MD and NMD, which improves discrimination between alternatives.

The primary motivations of this article are given as follows:

(1) To construct a novel structure of ff'-fractional fuzzy set (ff'-FFSs) that offers high flexibility in real-life circumstances. The idea of (ff'-FFSs) is an extension of the FFSs that are now in use since they can accommodate a lot of ambiguous data and give decision-making experts in MD and NMD an extra room.

(2) To develop a series of aggregation operators, which include ff'-fractional fuzzy arithmetic mean (ff'-FFAM), ff'-fractional fuzzy weighted averaging (ff'-FFWA), ff'-fractional fuzzy geometric mean (ff'-FFGM) and ff'-fractional fuzzy weighted geometric (ff'-FFWG) operators for aggregating the expert information in the context of ff'-FFSs.

(3) In MCDM studies, selecting a Ph.D. supervisor approach is crucial since it might negatively influence the future result of a student. The proposed work presents a ff'-FF CODAS technique for ranking alternatives in the context of ff'-FFS.

1.4. Objective of the study

In this article, we develop a CODAS model using ff'-fractional fuzzy (ff'-FF) information, inspired by the technique's particular decision-making capabilities. This study aims to achieve the following objectives of the current research:

(1) We establish a new idea of ff'-fractional fuzzy sets (ff'-FFSs), which is an extension of FFSs and is a broader in modelling uncertainty and hesitancy in expert information.

(2) We extend the traditional CODAS method to ff'-fractional fuzzy CODAS (ff'-FF CODAS) method, which a broader in MCMD environments.

(3) We employ the proposed ff'-FF CODAS method to apply our proven approach to a practical scenario. We have applied the design strategy to identify the best Ph.D. supervisor based on recommended criteria.

(4) We compared our proposed ff'-FF CODAS method with various existing techniques and operators to verify its success and stability.

(5) We conducted a sensitivity analysis based on two parameters f and f', to assess the ranking stability and robustness of the proposed method.

1.5. Novelties of the study

In the present study, we expanded the ideas of fractional fuzzy sets to ff'-fractional fuzzy sets (ff'-FFSs) as well as implemented it to the CODAS approach for choosing the most suitable Ph.D. supervisor in order to fulfil the aforementioned goals. A unique mathematical technique for expressing the meaning of ambiguous values, the CODAS approach within the mixed framework of fractional fuzzy sets, assists in overcoming the shortcomings of the current instruments. In addition, the CODAS approach offers a few noteworthy characteristics, such as being simpler, producing stable outcomes, and requiring a shorter duration for very extensive computations in math. The following is an overview of the key novelties of the proposed work.

(1) Throughout the framework of the ff'-FFSs, our aim for this research is to create a CODAS approach. With the MCDM scenarios that arose, we were able to effectively create a unique CODAS approach. We included relevant examples to further explain each step of the CODAS approach. The CODAS approach offers a number of noteworthy qualities, such as shorter computing times, more precise computational efficacy, and consistent outcomes.

(2) We develop a series of aggregation operators in the context of ff'-FF information, investigate their fundamental properties, and employ them in MCDM problem.

(3) We provided a visual illustration of the proposed model to ensure clarity and comprehensiveness.

(4) We demonstrate the usefulness of our proposed model by solving a real-world example. We used the proposed method for selecting an ideal Ph.D. supervisor plan in an educational field. The CODAS approach is still not being

investigated for selecting Ph.D. supervisors. Therefore, we applied the proposed model for the respective case study. We compare our developed method with existing methods and aggregation operators to demonstrate its usefulness and resilience.

Fig. 1 represents the graphical view of supervisor selection.

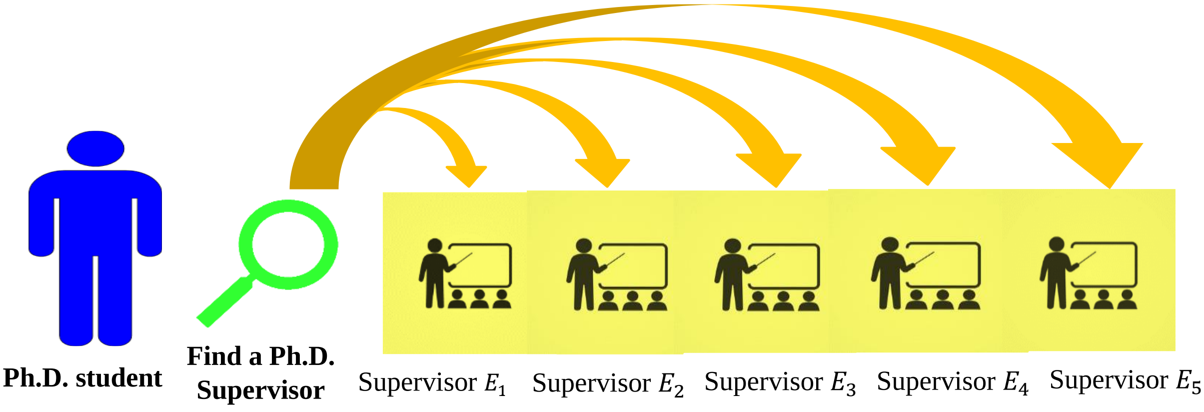

Figure 1: The complete process of Ph. D supervisor selection

1.6. Structure of the study

This article is structured as follows:: Section 2 outlines the fundamental concepts connected with FSs, IFSs, PFS, q-ROFS and FFSs. The new concept of the ff'-fractional fuzzy sets (ff'-FFSs) and operating principles that may be helpful in the decision-making process are covered in Section 3. Section 4 introduces aggregation operators for ff'-FFSs, including the ff'--FFAM (ff'-fractional fuzzy arithmetic mean), ff'-FFGM (ff'-fractional fuzzy geometric mean), ff'-FFWA (ff'-fractional fuzzy weighted averaging), ff'--FFWG (ff'--fractional fuzzy weighted geometric) aggregation operators. Section 5 provides a step-by-step explanation of the proposed ff'--FF CODAS method using ff'-fractional fuzzy information. Section 6 outlines a numerical example of the Ph.D. supervisor selection plan. Section 7 provides the numerical outcomes of the proposed method for the proposed case study. Section 8 offers a comparative analysis of the proposed method with the existing methods and operators. Section 9 demonstrates the sensitivity analysis based on the parameters $f$and f'.The conclusion of this research is given in Section 10.

Table 1 summarizes the abbreviations used in this work to assist viewers understand the vocabulary easily.

2. *Preliminaries*

In this section, we discuss the basic concepts, which will help us in developing the novel notions in upcoming sections. These concepts include fuzzy sets (FS), intuitionistic fuzzy sets (IFS), Pythagorean fuzzy sets (PFS), q-rung orthopair fuzzy sets (q-ROFS), and fractional fuzzy sets (FFS).

Definition 1: Consider the universe of discourse $\mathfrak{D} \neq \phi$. For every $x \in \mathfrak{D}$, a fuzzy set (FS) [7] Ã is the set which is mathematically represented as follows:

$$\tilde{A} = \{\langle x , \mu_{\tilde{A}}(x)\rangle | \, x \in \mathfrak{D}\}$$

Here $\mu_{\tilde{A}} : \mathfrak{D} \longrightarrow [0,1]$ represents the MD, which belongs to $[0,1]$ for each $x \in \mathfrak{D}$.

Definition 2: Consider the universe of discourse $\mathfrak{D} \neq \phi$. For every $x \in \mathfrak{D}$, the intuitionistic fuzzy set (IFS) [8] I is the set which is mathematically defined as follows:

$$I = \{\langle x, \mu_I(x), \upsilon_I(x)\rangle | x \in \mathfrak{D}\},$$

In which $\mu_I : \mathfrak{D} \longrightarrow [0,1]$ is the MD and $\upsilon_I : \mathfrak{D} \longrightarrow [0,1]$ is the NMD, which satisfy the condition $0 \leq \mu_I(x) + \upsilon_I(x) \leq 1$ for each $x \in \mathfrak{D}$.

The indeterminacy function for IFS is defined as follows:

$$\pi_I(x) = 1 - \mu_I(x) - \upsilon_I(x).$$

Definition 3: Consider the universe of discourse $\mathfrak{D} \neq \phi$. For every $x \in \mathfrak{D}$, the pythagorean fuzzy set (PFS) [10] P is the set which is mathematically defined as follows:

$$P = \{\langle x, \mu_P(x), \upsilon_P(x)\rangle | x \in \mathfrak{D}\},$$

In which $\mu_P : \mathfrak{D} \longrightarrow [0,1]$ is the MD and $\upsilon_P : \mathfrak{D} \longrightarrow [0,1]$ is the NMD, which satisfy the condition $0 \leq \left(\mu_P(x)\right)^2 + \left(\upsilon_P(x)\right)^2 \leq 1$ for each $x \in \mathfrak{D}$. The indeterminacy function for PFS is defined as follows:

$$\pi_P(x) = \left(1 - \left(\mu_P(x)\right)^2 - \left(\upsilon_P(x)\right)^2\right)^{\frac{1}{2}}.$$

Table 1: Abbreviations and their meanings

Abbreviations	**Meanings**
FS	Fuzzy set
MCDM	Multi-criteria decision-making
MCGDM	Multi-criteria group decision-making
IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
q-ROFS	q-rung orthopair fuzzy set
FFS	Fractional fuzzy set
ff'-FF CODAS	ff'-Fractional fuzzy Combinative Distance-Based Assessment
HD/ED	Hamming distance/Euclidean distance
MD/NMD	Membership degree/Non-membership degree
WSM/WPM	Weighted sum model/Weighted product model
WASPAS	Weighted assessment sum and product method
multi-MOORA	multi-objective optimization ratio analysis
ff'-FFSs	ff'-fractional fuzzy sets
ff'-FFN	ff'-fractional fuzzy number
ff'-FFAM	ff'-fractional fuzzy arithmetic mean
ff'-FFWA	ff'-fractional fuzzy weighted averaging
ff'-FFGM	ff'-fractional fuzzy geometric mean
ff'-FFWG	ff'-fractional fuzzy weighted geometric
DM	Decision-making

Definition 4: Consider the universe of discourse $\mathfrak{D} \neq \phi$. For every $x \in \mathfrak{D}$, a q-ROFS [11] Q is a set which is mathematically defined as follow:

$$Q = \{\langle x, \mu_Q(x), \upsilon_Q(x)\rangle | x \in \mathfrak{D}\},$$

where $\mu_Q : \mathfrak{D} \longrightarrow [0,1]$ the MD and $\upsilon_Q : \mathfrak{D} \longrightarrow [0,1]$ is the NMD, which satisfy the condition $0 \leq (\mu_Q(x))^q + (\upsilon_Q(x))^q \leq 1$ for each $x \in \mathfrak{D}$. The indeterminacy function for (q-ROFS) is defined as follows:

$$\pi_Q(x) = \left(1 - (\mu_Q(x))^q - (\upsilon_Q(x))^q\right)^{\frac{1}{q}}.$$

Definition 5: Consider the universe of discourse $\mathfrak{D} \neq \phi$. For every $x \in \mathfrak{D}$, a fractional fuzzy set (FFS) [12] $\digamma$ is a set which mathematically defined as follows:

$$\digamma = \{\langle x, \mu_\digamma(x), \upsilon_\digamma(x)\rangle | x \in \mathfrak{D}\},$$

where $\mu_\digamma(x) : \mathfrak{D} \longrightarrow [0,1]$ the MD and $\upsilon_\digamma(x) : \mathfrak{D} \longrightarrow [0,1]$ is the NMD, which satisfies the condition $0 \leq (\mu_\digamma(x))^f + (\upsilon_\digamma(x))^f \leq 1$ where $f = \frac{p}{q} \geq 1$ for each $x \in \mathfrak{D}$. The indeterminacy function for FFS is defined as follows:

$$\pi_\digamma(x) = \left(1 - (\mu_\digamma(x))^f - (\upsilon_\digamma(x))^f\right)^{\frac{1}{f}}.$$

It is clear from the definitions mentioned above that FFS is perhaps the most often used kind among q-ROFS, IFS, and PFS satisfying the condition $0 \leq (\mu_\digamma(x))^f + (\upsilon_\digamma(x))^f \leq 1$, where $f = \frac{p}{q} \geq 1$.

The extension of various fuzzy sets is shown in the following Fig. 2.

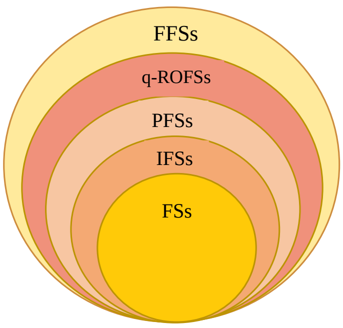

Figure 2: Fuzzy set and several of its extensions

3. Construction Of New Idea Of ff'-Fractional Fuzzy Sets

This section presents a fresh idea by extending the notion of fractional fuzzy sets to ff'-fractional fuzzy sets (ff'-FFSs), which serve as effective and generalized tools for addressing uncertainty in decision-making frameworks.

Definition 6: Consider the universe of discourse $\mathfrak{D} \neq \phi$. For every $x \in \mathfrak{D}$. Then ff'-fractional fuzzy set (ff'-FFSs) $\mathcal{F}$ is a set which mathematically is defined as follows:

$$\mathcal{F} = \{\langle x, \mu_{\mathcal{F}}(x), \upsilon_{\mathcal{F}}(x)\rangle | x \in \mathfrak{D}\},$$

where $\mu_{\mathcal{F}}(x) : \mathfrak{D} \longrightarrow [0,1]$ is the membership grade (MG) and $\upsilon_{\mathcal{F}}(x) : \mathfrak{D} \longrightarrow [0,1]$ is the non-membership grade (NMG) which fulfill the requirements $0 \leq \left(\mu_{\mathcal{F}}(x)\right)^{f} + (\upsilon_{\mathcal{F}}(x))^{f'} \leq 1$ and $\left((f \neq f') = p/q \geq 1\right)$.

As we know that f and f' are any fractional values of the form $(f \neq f') = \frac{p}{q}, q \neq 0$, it is obvious that for any $x \in \mathfrak{D}$, there exists f and f' such that

$$0 \leq \left(\mu_{\mathcal{F}}(x)\right)^{f} \leq 1 \text{ and } 0 \leq (\upsilon_{\mathcal{F}}(x))^{f'} \leq 1.$$

Then, we have

$$0 \leq \left(\mu_{\mathcal{F}}(x)\right)^{f} + \left(\upsilon_{\mathcal{F}}(x)\right)^{f'} \leq 1.$$

Example 1: Suppose that $\mathfrak{D} = \{x_1, x_2, x_3, x_4, x_5\}$ is a universe of discourse. Then a collection $\mathcal{F}$ over $\mathfrak{D}$ for $f = 8/3$ and $f' = 5/2$ is given below:

$$\mathcal{F} = \{(x_1, 0.63, 0.53), (x_2, 0.65, 0.55), (x_3, 0.67, 0.57), (x_4, 0.69, 0.59), (x_5, 0.71, 0.61)\}$$

Hence $\mathcal{F}$ is a ff'-FFS. Because we see that

$$0 \leq 0.63 \leq 1, 0 \leq 0.53 \leq 1 \text{ and } (0.63)^{8/3} + (0.53)^{5/2} = 0.4961.$$
$$0 \leq 0.65 \leq 1, 0 \leq 0.55 \leq 1 \text{ and } (0.65)^{8/3} + (0.55)^{5/2} = 0.5413.$$
$$0 \leq 0.67 \leq 1, 0 \leq 0.57 \leq 1 \text{ and } (0.67)^{8/3} + (0.57)^{5/2} = 0.5890.$$
$$0 \leq 0.69 \leq 1, 0 \leq 0.59 \leq 1 \text{ and } (0.69)^{8/3} + (0.59)^{5/2} = 0.6391.$$
$$0 \leq 0.71 \leq 1, 0 \leq 0.61 \leq 1 \text{ and } (0.67)^{8/3} + (0.57)^{5/2} = 0.6918.$$

The above example clarifies that q-ROFS is defined only for positive integers ($\text{i.e.}, q \geq 1$). However, there exist a lot of values between any two positive integers. Therefore, ff'-FFS has the ability to discuss all these points that exist between any two positive integers, which give a lot of room to the experts in the process of decision-making.

Theoretically, enabling different fractional parameters $f \neq f'$ allows experts to be flexible when stating asymmetric confidence in the effects of the MD and NMD. In contrast to classical fractional fuzzy sets, where $f \neq f'$ imposes the same level of weighting, the ff'-FFS framework allows the consideration of the influence of positive and negative evidence independently. An example of this is that in the case of assessing candidates for a scholarship, an expert can be more certain concerning evidence favoring the eligibility (MD) of a candidate compared to evidence opposing it (NMD); applying $f \neq f'$ allows the model to capture this asymmetry of confidence, resulting in more accurate and precise decision-making.

In case of $f = f'$ and for various inputs of p along with q, we get various types of FSs.

(a) When $\left(f = f' = \frac{p}{q} \geq 1, p = q\right)$, then the ff'-FFS reduces to intuitionistic fuzzy sets.

(b) When $\left(f = f' = \frac{p}{q} \geq 1, p = 2q\right)$, then the ff'-FFS reduces to Pythagorean fuzzy sets.

(c) When $\left(f = f' = \frac{p}{q} \geq 1, p = 3q\right)$, then the ff'-FFS reduces to Fermatean fuzzy sets.

(d) When $\left(f = f' = \frac{p}{q} \geq 1, p \geq 3, q = 1\right)$, then the ff'-FFS reduces to q-rung orthopair fuzzy sets.

(e) When $\left(f = f' = \frac{p}{q} \geq 1\right)$, then the ff'-FFS reduces to fractional fuzzy sets.

In simple way, we can represent a ff'-fractional fuzzy number (ff'-FFN) as $\mathcal{F}_{\dot{g}} = \left(\mu_{\dot{g}}, \upsilon_{\dot{g}}\right)$ $(\dot{g} = 1, 2, \ldots, n)$.

Definition 7: Consider $\mathcal{F}_{\dot{g}} = \left(\mu_{\dot{g}}, \upsilon_{\dot{g}}\right)$ $(\dot{g} = 1,2, \ldots, n)$ be a collection of ff'-FFNs, then the indeterminacy function for ff'-FFNs is defined as,

$$\pi_{\mathcal{F}}(x) = \left(1 - \left(\mu_{\mathcal{F}}(x)\right)^{f} - \left(\upsilon_{\mathcal{F}}(x)\right)^{f'}\right)^{\frac{1}{ff'}}$$

Definition 8: Let $\mathcal{F}_1 = (\mu_1, \upsilon_1)$ and $\mathcal{F}_2 = (\mu_2, \upsilon_2)$ be two ff'-fractional fuzzy numbers ff'-FFNs. Then some basic properties of ff'-FFS are defined as follows:

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $\mu_1 < \mu_2$ and $\upsilon_1 > \upsilon_2$;

(2) $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{F}_1 \supseteq \mathcal{F}_2$;

(3) $\mathcal{F}_1 \cup \mathcal{F}_2 = (max(\mu_1, \mu_2), \min(\upsilon_1, \upsilon_2))$;

(4) $\mathcal{F}_1 \cap \mathcal{F}_2 = \left(\min(\mu_1, \mu_2), max(\upsilon_1, \upsilon_2)\right)$;

(5) $\mathcal{F}_1^c = (\upsilon_1, \mu_1)$.

To compare two ff'-FFNs, we use the score and accuracy functions. The following is the definition of the score and accuracy functions of the ff'-FFNs.

Definition 9: Consider $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\dots,n)$ be a collection of ff'-FFNs, then the score as well as accuracy functions are defined as:

$$\dot{S}\dot{r} = \mu_{\dot{g}}{}^{f} - \upsilon_{\dot{g}}{}^{f'} \in [-1,1].$$
$$\breve{A}\mathfrak{c}\mathfrak{c} = \mu_{\dot{g}}{}^{f} + \upsilon_{\dot{g}}{}^{f'} \in [0,1].$$

Definition 10: Let $\mathcal{F}_1 = (\mu_1, \upsilon_1)$ and $\mathcal{F}_2 = (\mu_2, \upsilon_2)$ be two ff'-FFNs and let $\dot{S}\dot{r}(\mathcal{F}_{\dot{g}})$ and $\breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_{\dot{g}})$ $(\dot{g} = 1,2)$ be its corresponding score and accuracy values, then

a) $\dot{S}\dot{r}(\mathcal{F}_1) \prec \dot{S}\dot{r}(\mathcal{F}_2) \Rightarrow \mathcal{F}_1 \prec \mathcal{F}_2$;
b) $\dot{S}\dot{r}(\mathcal{F}_1) \succ \dot{S}\dot{r}(\mathcal{F}_2) \Rightarrow \mathcal{F}_1 \succ \mathcal{F}_2$;
c) $\dot{S}\dot{r}(\mathcal{F}_1) = \dot{S}\dot{r}(\mathcal{F}_2)$, then
   i. $\breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_1) \prec \breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_2) \Rightarrow \mathcal{F}_1 \prec \mathcal{F}_2$;
   ii. $\breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_1) \succ \breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_2) \Rightarrow \mathcal{F}_1 \succ \mathcal{F}_2$;
   iii. $\breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_1) = \breve{A}\mathfrak{c}\mathfrak{c}(\mathcal{F}_2) \Rightarrow \mathcal{F}_1 = \mathcal{F}_2$.

Next, we define some basic operations for ff'-FFNs.

Definition 11: Let $\mathcal{F}_1 = (\mu_1, \upsilon_1)$ and $\mathcal{F}_2 = (\mu_2, \upsilon_2)$ be two ff'-FFNs and $\xi > 0$ represents any real numbers, then the operational rules for ff'-FFNs are specified as under:

1) $$\mathcal{F}_1 \oplus \mathcal{F}_2 = \left\langle ((\mu_1)^f + (\mu_2)^f - (\mu_1)^f \cdot (\mu_2)^f)^{\frac{1}{f}}, ((\upsilon_1)^{f'} \cdot (\upsilon_2)^{f'})^{\frac{1}{f'}} \right\rangle. \quad (1)$$

2) $$\mathcal{F}_1 \otimes \mathcal{F}_2 = \left\langle ((\mu_1)^f \cdot (\mu_2)^f)^{\frac{1}{f}}, ((\upsilon_1)^{f'} + (\upsilon_2)^{f'} - (\upsilon_1)^{f'} \cdot (\upsilon_1)^{f'})^{\frac{1}{f'}} \right\rangle. \quad (2)$$

3) $$\xi \cdot \mathcal{F}_1 = \left\langle (1 - (1 - (\mu_1)^f)^{\xi})^{\frac{1}{f}}, \left(\left((\upsilon_1)^{f'}\right)^{\xi}\right)^{\frac{1}{f'}} \right\rangle. \quad (3)$$

4) $$\mathcal{F}_1{}^{\xi} = \left\langle (((\mu_1)^f)^{\xi})^{\frac{1}{f}}, \left(1 - \left(1 - (\upsilon_1)^{f'}\right)^{\xi}\right)^{\frac{1}{f'}} \right\rangle. \quad (4)$$

Example 2: Let $\mathcal{F}_1 = \langle 0.6, 0.4\rangle, \mathcal{F}_1 = \langle 0.7, 0.5\rangle$ and $\xi = 0.5$ represents two ff'-FFNs suppose that $f = \frac{4}{3}$ and $f' = \frac{5}{4}$ are two fractional values, we have

1) $$\mathcal{F}_1 \oplus \mathcal{F}_2 = \left\langle \left((0.6)^{\frac{4}{3}} + (0.7)^{\frac{4}{3}} - (0.6)^{\frac{4}{3}} \cdot (0.7)^{\frac{4}{3}}\right)^{\frac{3}{4}}, \left((0.4)^{\frac{5}{4}} \cdot (0.5)^{\frac{5}{4}}\right)^{\frac{4}{5}} \right\rangle = \langle 0.8562, 0.2000\rangle.$$

2) $$\mathcal{F}_1 \otimes \mathcal{F}_2 = \left\langle \left((0.6)^{\frac{4}{3}} \cdot (0.7)^{\frac{4}{3}}\right)^{\frac{3}{4}}, \left((0.4)^{\frac{5}{4}} + (0.5)^{\frac{5}{4}} - (0.4)^{\frac{5}{4}} \cdot (0.5)^{\frac{5}{4}}\right)^{\frac{4}{5}} \right\rangle = \langle 0.4200, 0.6688\rangle.$$

3) $$0.5 \cdot \mathcal{F}_1 = \left\langle \left(1 - \left(1 - (0.6)^{\frac{4}{3}}\right)^{0.5}\right)^{\frac{3}{4}}, \left(\left((0.4)^{\frac{5}{4}}\right)^{0.5}\right)^{\frac{4}{5}} \right\rangle = \langle 0.4025, 0.6324\rangle.$$

4) $$\mathcal{F}_1{}^{0.5} = \left\langle \left(\left((0.6)^{\frac{4}{3}}\right)^{0.5}\right)^{\frac{3}{4}}, \left(1 - \left(1 - (0.4)^{\frac{5}{4}}\right)^{0.5}\right)^{\frac{4}{5}} \right\rangle = \langle 0.7746, 0.2471\rangle.$$

Theorem 1: Assume that $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,3)$ be any three ff'-FFNs with $\lambda \geq 0$. Then the following identities hold.

(1) $\mathcal{F}_1 \oplus \mathcal{F}_2 = \mathcal{F}_2 \oplus \mathcal{F}_1$;
(2) $\mathcal{F}_1 \otimes \mathcal{F}_2 = \mathcal{F}_2 \otimes \mathcal{F}_1$;
(3) $(\mathcal{F}_1 \oplus \mathcal{F}_2) \oplus \mathcal{F}_3 = \mathcal{F}_1 \oplus (\mathcal{F}_2 \oplus \mathcal{F}_3)$;
(4) $(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3 = \mathcal{F}_1 \otimes (\mathcal{F}_2 \otimes \mathcal{F}_3)$;
(5) $\lambda \mathcal{F}_1 \otimes \lambda \mathcal{F}_2 = \lambda(\mathcal{F}_1 \otimes \mathcal{F}_2),\ \lambda \geq 0$;
(6) $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{\lambda} = \mathcal{F}_1^{\lambda} \otimes \mathcal{F}_2^{\lambda},\ \lambda \geq 0$;
(7) $\mathcal{F}_1^{\lambda_1} \otimes \mathcal{F}_1^{\lambda_2} = (\mathcal{F}_1)^{\lambda_1 \oplus \lambda_2}, \lambda_1 \geq 0, \lambda_2 \geq 0$;

Proof: We will prove only (1) and (2), as the remaining are similar to that.

(1) To show that $\mathcal{F}_1 \oplus \mathcal{F}_2 = \mathcal{F}_2 \oplus \mathcal{F}_1$, we have

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = \left\langle ((\mu_1)^f + (\mu_2)^f - (\mu_1)^f \cdot (\mu_2)^f)^{\frac{1}{f}}, \left((\upsilon_1)^{f'} \cdot (\upsilon_2)^{f'}\right)^{\frac{1}{f'}} \right\rangle.$$

$$= \left\langle ((\mu_2)^f + (\mu_1)^f - (\mu_2)^f \cdot (\mu_1)^f)^{\frac{1}{f}}, ((\upsilon_2)^{f'} \cdot (\upsilon_1)^{f'})^{\frac{1}{f'}} \right\rangle$$
$$= \mathcal{F}_2 \oplus \mathcal{F}_1.$$

(2) Here, we need to show that $\mathcal{F}_1 \otimes \mathcal{F}_2 = \mathcal{F}_2 \otimes \mathcal{F}_1$, so we have

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \left\langle ((\mu_1)^f \cdot (\mu_2)^f)^{\frac{1}{f}}, ((\upsilon_1)^{f'} + (\upsilon_2)^{f'} - (\upsilon_1)^{f'} \cdot (\upsilon_1)^{f'})^{\frac{1}{f'}} \right\rangle$$
$$= \left\langle ((\mu_2)^f \cdot (\mu_1)^f)^{\frac{1}{f}}, ((\upsilon_2)^{f'} + (\upsilon_1)^{f'} - (\upsilon_2)^{f'} \cdot (\upsilon_1)^{f'})^{\frac{1}{f'}} \right\rangle$$
$$= \mathcal{F}_2 \otimes \mathcal{F}_1.$$

4. *Aggregation Operators for ff'-Fractional Fuzzy Set*

In this section, we discuss a set of aggregation operators based on ff'-fractional fuzzy information. These include ff'-fractional fuzzy arithmetic mean (ff'-FFAM), ff'-fractional fuzzy weighted averaging (ff'-FFWA), ff'-fractional fuzzy geometric mean (ff'-FFGM), and ff'-fractional fuzzy weighted geometric (ff'-FFWG) operators. We will also discuss several of their important properties.

Definition 12: Consider $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\dots,n)$ represents a collection of ff'-FFNs. Then the ff'-fractional fuzzy arithmetic mean (ff'-FFAM) operator can be defined as follows:

$ff' -$ FFAM $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) = \oplus_{\dot{g}=1}^{n} \mathcal{F}_{\dot{g}}$

$$= \left\langle \left(1 - \prod_{\dot{g}=1}^{n}\left(1 - (\mu_{\dot{g}})^f\right)\right)^{\frac{1}{f}}, \left(\prod_{\dot{g}=1}^{n}(\upsilon_{\dot{g}})^{f'}\right)^{\frac{1}{f'}} \right\rangle. \quad (5)$$

Theorem 2: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\dots,n)$ be a collection of ff'-FFNs. Then the (ff'-FFAM) satisfy the following properties.

(1) **(Property of Idempotency)**: Assume that each $\mathcal{F}_{\dot{g}}$ $(\dot{g} = 1,2,\dots,n)$ is the same that is $\mathcal{F}_{\dot{g}} = \mathcal{F}$. Then
$$ff' - \text{FFAM}\ (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) = \mathcal{F}. \quad (6)$$

(2) **(Property of Boundedness):** Let $\mathcal{F}_{\dot{g}}^{-} = min_{1 \le \dot{g} \le n}\{\mathcal{F}_{\dot{g}}\}$ and $\mathcal{F}_{\dot{g}}^{+} = max_{1 \le \dot{g} \le n}\{\mathcal{F}_{\dot{g}}\}$ be two distinct collections of ff'-FFNs. Then we have
$$\mathcal{F}_{\dot{g}}^{-} \le ff' - \text{FFAM}\ (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) \le \mathcal{F}_{\dot{g}}^{+}. \quad (7)$$

(3) **(Property of Monotonicity)**: Let $\mathcal{F}_{\dot{g}}$ and $\mathcal{F}_{\dot{g}}^{*}$ be two different collections of ff'-FFNs with $\mathcal{F}_{\dot{g}} \le \mathcal{F}_{\dot{g}}^{*}$ for all $\dot{g}$. Then
$$ff' - \text{FFAM}\ (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) \le ff' - \text{FFAM}\ (\mathcal{F}_1^{*}, \mathcal{F}_2^{*}, \dots, \mathcal{F}_n^{*}). \quad (8)$$

Next, we define the ff'-fractional fuzzy weighted averaging (ff'-FFWA) operator.

Definition 13: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\dots,n)$ be a collection of ff'-FFNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ represents its weighted vector such that $\omega_{\dot{g}} \ge 0$ with $\sum_{\dot{g}=1}^{n} \omega_{\dot{g}} = 1$. Based on operating rules defined in Definition 10, the ff'-fractional fuzzy weighted averaging (ff'-FFWA) operator can be defined by the following equation.

$ff' -$ FFWA $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) = \oplus_{\dot{g}=1}^{n} (\omega_{\dot{g}} \cdot \mathcal{F}_{\dot{g}})$

$$= \left\langle \left(1 - \prod_{\dot{g}=1}^{n}\left(1 - (\mu_{\dot{g}})^f\right)^{\omega_{\dot{g}}}\right)^{\frac{1}{f}}, \left(\prod_{\dot{g}=1}^{n}(\upsilon_{\dot{g}})^{f' \cdot \omega_{\dot{g}}}\right)^{\frac{1}{f'}} \right\rangle \quad (9)$$

Theorem 3: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\dots,n)$ be a collection of ff' −FFNs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ represents its weight vectors such that $\omega_{\dot{g}} \ge 0$ with $\sum_{\dot{g}=1}^{n} \omega_{\dot{g}} = 1$. The following properties hold for (ff'-FFWA) operator.

(1) **(Property of Idempotency)**. Assume that each $\mathcal{F}_{\dot{g}}$ $(\dot{g} = 1,2,\dots,n)$ is the same that is $\mathcal{F}_{\dot{g}} = \mathcal{F}$. Then
$$ff' - \text{FFWA}\ (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) = \mathcal{F}. \quad (10)$$

(2) **(Property of Boundedness).** Assume that $\mathcal{F}_{\dot{g}}^{-} = min_{1 \le \dot{g} \le n}\{\mathcal{F}_{\dot{g}}\}$ as well as $\mathcal{F}_{\dot{g}}^{+} = max_{1 \le \dot{g} \le n}\{\mathcal{F}_{\dot{g}}\}$. Then
$$\mathcal{F}_{\dot{g}}^{-} \le ff' - \text{FFWA}\ (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) \le \mathcal{F}_{\dot{g}}^{+}. \quad (11)$$

(3) **(Property of Monotonicity)**. Let $\mathcal{F}_{\dot{g}}$ and $\mathcal{F}_{\dot{g}}^{*}$ be two collection of ff'-FFNs with $\mathcal{F}_{\dot{g}} \le \mathcal{F}_{\dot{g}}^{*}$ for every $\dot{g}$. Then
$$ff' - \text{FFWA}\ (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) \le ff' - \text{FFWA}\ (\mathcal{F}_1^{*}, \mathcal{F}_2^{*}, \dots, \mathcal{F}_n^{*}). \quad (12)$$

Definition 14: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\dots,n)$ be a collection of ff'-FFNs. Then the ff'-fractional fuzzy geometric mean (ff' −FFGM) operator can be defined as follows;

$ff' -$ FFGM $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n) = \otimes_{\dot{g}=1}^{n} \mathcal{F}_{\dot{g}}$

$$= \left\langle \left(\prod_{\dot{g}=1}^{n}\left((\mu_{\dot{g}})^f\right)\right)^{\frac{1}{f}}, \left(1 - \prod_{\dot{g}=1}^{n}\left(1 - (\upsilon_{\dot{g}})^{f'}\right)\right)^{\frac{1}{f'}} \right\rangle. \quad (13)$$

Theorem 4: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\ldots,n)$ be a collection of ff'-FFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ represents its weight vectors such that $\omega_{\dot{g}} \geq 0$ with $\sum_{\dot{g}=1}^{n} \omega_{\dot{g}} = 1$. Then the following properties hold for ff'-FFGM operator.

(1) **(Property of Idempotency)**: Assume that all $\mathcal{F}_{\dot{g}}$ $(\dot{g} = 1,2,\ldots,n)$ are equal such that $\mathcal{F}_{\dot{g}} = \mathcal{F}$. Then

$$ff' - \text{FFGM}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = \mathcal{F}. \tag{14}$$

(2) **(Property of Boundedness):** Assume that $\mathcal{F}_{\dot{g}}^{-} = min_{1\leq\dot{g}\leq n}\{\mathcal{F}_{\dot{g}}\}$ as well as $\mathcal{F}_{\dot{g}}^{+} = max_{1\leq\dot{g}\leq n}\{\mathcal{F}_{\dot{g}}\}$. Then

$$\mathcal{F}_{\dot{g}}^{-} \leq ff' - \text{FFGM}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) \leq \mathcal{F}_{\dot{g}}^{+}. \tag{15}$$

(3) **(Property of Monotonicity)**: Assume that $\mathcal{F}_{\dot{g}}$ and $\mathcal{F}_{\dot{g}}^{*}$ be two collection of $(ff'$ −FFNs with $\mathcal{F}_{\dot{g}} \leq \mathcal{F}_{\dot{g}}^{*}$ for every $\dot{g}$. Then

$$ff' - \text{FFGM}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) \leq ff' - \text{FFGM}(\mathcal{F}_1^{*}, \mathcal{F}_2^{*}, \ldots, \mathcal{F}_n^{*}). \tag{16}$$

Definition 15: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\ldots,n)$ be a collection of ff'-FFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is their associated weighted vector such that $\omega_{\dot{g}} \geq 0$ with $\sum_{\dot{g}=1}^{n} \omega_{\dot{g}} = 1$. Then the ff'-fractional fuzzy weighted geometric (ff'-FFWG) operator can be defined by as follows:

$$ff' - \text{FFWG}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = \otimes_{\dot{g}=1}^{n} (\mathcal{F}_{\dot{g}})^{\omega_{\dot{g}}}$$
$$= \left\langle \left(\prod_{\dot{g}=1}^{n} \left((\mu_{\dot{g}})^{f} \right)^{\omega_{\dot{g}}} \right)^{\frac{1}{f}}, \left(1 - \prod_{\dot{g}=1}^{n} \left(1 - (\upsilon_{\dot{g}})^{f'} \right)^{\omega_{\dot{g}}} \right)^{\frac{1}{f'}} \right\rangle. \tag{17}$$

Theorem 5: Let $\mathcal{F}_{\dot{g}} = (\mu_{\dot{g}}, \upsilon_{\dot{g}})$ $(\dot{g} = 1,2,\ldots,n)$ be a collection of ff'-FFNs along with $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ represents its weight vectors such that $\omega_{\dot{g}} \geq 0$ with $\sum_{\dot{g}=1}^{n} \omega_{\dot{g}} = 1$. The following properties hold for (ff'-FFWG) operator.

(1) **(Property of Idempotency)**: Let all $\mathcal{F}_{\dot{g}}$ $(\dot{g} = 1,2,\ldots,n)$ are the same that is $\mathcal{F}_{\dot{g}} = \mathcal{F}$. Then

$$ff' - \text{FFWG}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = \mathcal{F}. \tag{18}$$

(2) **(Property of Boundedness):** Assume that $\mathcal{F}_{\dot{g}}^{-} = min_{1\leq\dot{g}\leq n}\{\mathcal{F}_{\dot{g}}\}$ as well as $\mathcal{F}_{\dot{g}}^{+} = max_{1\leq\dot{g}\leq n}\{\mathcal{F}_{\dot{g}}\}$. Then

$$\mathcal{F}_{\dot{g}}^{-} \leq ff' - \text{FFWG}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) \leq \mathcal{F}_{\dot{g}}^{+}. \tag{19}$$

(3) **(Property of Monotonicity):** Assume that $\mathcal{F}_{\dot{g}}$ and $\mathcal{F}_{\dot{g}}^{*}$ be two collection of ff'-FFNs with $\mathcal{F}_{\dot{g}} \leq \mathcal{F}_{\dot{g}}^{*}$ for every $\dot{g}$. Then

$$ff' - \text{FFWG}\,(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) \leq ff' - \text{FFWG}\,(\mathcal{F}_1^{*}, \mathcal{F}_2^{*}, \ldots, \mathcal{F}_n^{*}). \tag{20}$$

5. *The Extended ff'- Fractional Fuzzy CODAS (ff'-FF CODAS) Method*

In this section, we discuss a novel ff'-fractional fuzzy-CODAS (ff'-FF CODAS) method and amplification of the CODAS approach involving ff'-fractional fuzzy inputs. The CODAS technique is an interesting and helpful MCDM technique that Keshavarz Ghorabaee et al. [15] just released. To calculate the negative ideal solution for ordering, this strategy employs the Euclidean along with taxicab distances for every alternative. It is preferable to choose an alternative that is located further away than all of them. In crisp surroundings, the Euclidean along with taxicab distances have been employed in the CODAS technique. However, Keshavarz Ghorabaee et al. [16] recommend using fuzzy-based Euclidean along with fuzzy-based Hamming measures in fuzzy challenges as opposed to crisp elements. ff'-fractional fuzzy-based Euclidean as well as Hamming distances, as laid out by Park et al. [34], are employed in step 5 for the above reason. Assume that we have n numbers of alternatives $\mathfrak{T} = \{\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_n\}$, m criterions $\mathfrak{S} = \{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m\}$ and l decision makers $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_l\}$ along with experts weights $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ such that $\omega_c \geq 0$ with $\sum_{\dot{g}=1}^{l} \omega_{\dot{g}} = 1$. The following sequence of steps outlines the various stages of the suggested procedure:

Step 1: In this step, we collect the information from $\dot{g}$ experts in the form of ff'-fractional fuzzy numbers (ff'-FFNs) to construct a decision-making matrix. The corresponding decision-making matrix ($\mathcal{D}_l$) is given as follows:

$$\mathcal{D}_l = (\mathcal{F}_{ij}^{l})_{n\times m} = \begin{bmatrix} \mathcal{F}_{11}^{l} & \mathcal{F}_{12}^{l} & \cdots & \mathcal{F}_{1m}^{l} \\ \mathcal{F}_{21}^{l} & \mathcal{F}_{22}^{l} & \cdots & \mathcal{F}_{2m}^{l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{F}_{n1}^{l} & \mathcal{F}_{n2}^{l} & \ldots & \mathcal{F}_{nm}^{l} \end{bmatrix}, \tag{21}$$

Where $\mathcal{F}_{ij}^{l}$ is the outcome grade of the ith alternative over jth criterion along with lth expert where $(i = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, m)$ and $(l = 1,2, \ldots, \dot{g})$. The known weight for every criterion that is obtained from the various experts is an additional consideration that requires to be made decided upon in the very beginning.

$$\omega = [\omega_{jl}]_{1\times m}. \tag{22}$$

Step 2: Aggregate the expert information into a single decision matrix ($\mathcal{D}$) and assess the combined weights of every criterion. Based on the ff'-FFWA operator defined in Eq. (9), this step is going to be completed.

$$\mathcal{D} = [\mathcal{F}_{ij}]_{n\times m} = \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & \cdots & \mathcal{F}_{1m} \\ \mathcal{F}_{21} & \mathcal{F}_{22} & \cdots & \mathcal{F}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{F}_{n1} & \mathcal{F}_{n1} & \dots & \mathcal{F}_{nm} \end{bmatrix} \tag{23}$$

Step 3: Compute a weighted normalized decision matrix. The following equations will be used to determine the weighted normalized productivity standards:

$$\mathfrak{R} = [\hat{r}_{ij}]_{n\times m};$$
$$\hat{r}_{ij} = \omega_j \otimes \mathcal{F}_{ij} \tag{24}$$

in which ω_j indicates the weights of the jth criterion.

Step 4: Figure out the negative-ideal solution (place) in the way that is described below:

$$\mathcal{NS} = [\mathcal{ns}_j]_{1\times m}$$
$$\mathcal{ns}_j = min_i\, \hat{r}_{ij}. \tag{25}$$

Step 5: Considering the negative ideal solution, determine the alternatives' normalized Hamming along with Euclidean distances (HD_i and ED_i, respectively) by utilizing the following formulae;

$$\text{HD}_i = \sqrt{\sum_{j=1}^{m} (\hat{r}_{ij} - \mathcal{ns}_j)^2} \tag{26}$$

$$\text{ED}_i = \sum_{j=1}^{m} |\hat{r}_{ij} - \mathcal{ns}_j|. \tag{27}$$

Step 6: Calculate the relative assessment matrix (RA), which is provided by the equation given below:

$$RA = [\mathcal{P}_{ik}]_{n\times n} \tag{28}$$

$$\mathcal{P}_{ik} = (\text{ED}_i - \text{ED}_k) + \Psi(\text{ED}_i - \text{ED}_k) \times (\text{HD}_i - \text{HD}_k). \tag{29}$$

We can describe Ψ, as threshold function. The design is used to understand the equivalence of the Euclidean distances between the two alternatives.

$$\Psi(x) = \begin{cases} 1, & if\ |x| \geq \gamma \\ 0, & if\ |x| < \gamma \end{cases}. \tag{30}$$

The expert may establish the threshold parameter in this function that is denoted by γ. The values for this parameter ought to be configured to a value that ranges from 0.01 to 0.05. Two alternatives are contrasted as well, employing the taxicab distance if the variation among their Euclidean distances is less extensive than γ. In the proposed work, we choose the value of $\gamma = 0.03$ for the purpose of calculations. The value of $\gamma = 0.03$ is chosen to provide an equal strength of discrimination between alternatives. Any small value would turn the method into an oversensitive one, and any large value would lower the discrimination of the alternatives that are closely evaluated. This option is what makes sure there is a good switch between Euclidean and Hamming distance assessment in the CODAS method and the rankings remain consistent and correct.

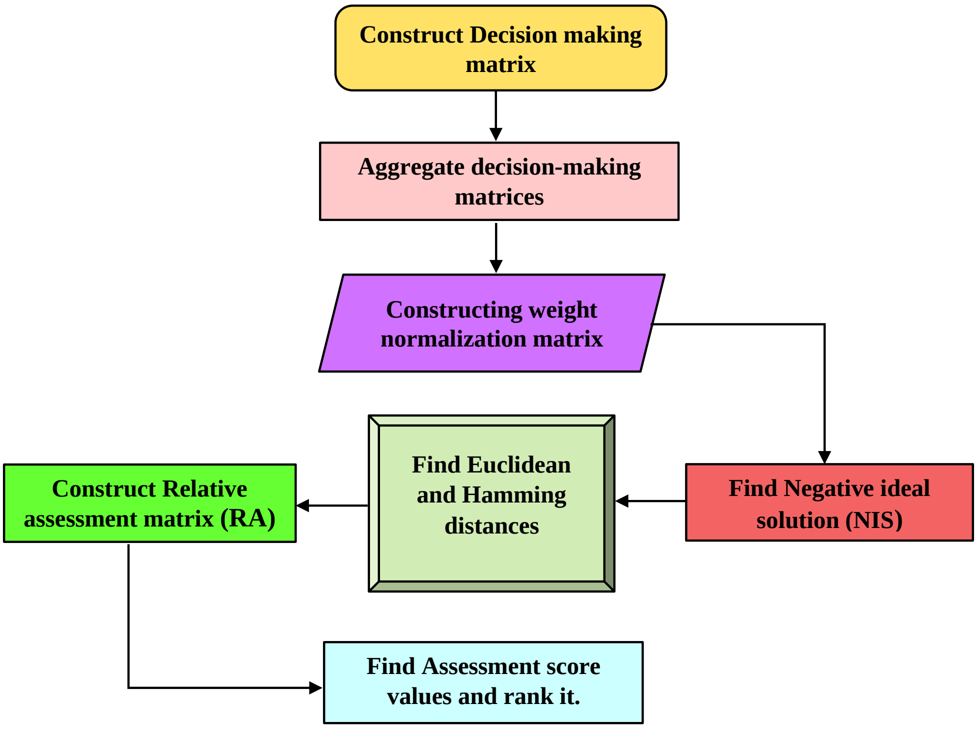

Figure 3: A graphical illustration of the proposed ff'-FFS CODAS model

Step 7: Calculate the assessment score ($\mathcal{AS}_i$) of every alternative, which is calculated by using the following formula:

$$\mathcal{AS}_i = \sum_{k=1}^{n} \mathcal{P}_{ik} \tag{31}$$

Step 8: Arrange the alternatives in a decreasing sequence based on the assessment score results. The greatest option is the one which has the greatest assessment score value.

Fig. 3 shows the graphical illustration of the proposed ff'-FF CODAS model.

6. Problem Statement

The problem of selecting a Ph.D. supervisor is especially a complex decision-making problem because it is a long-term MCDM problem that affects both the academic and professional life of a student. The choice entails non-quantifiable factors, including research compatibility, mentorship approach, and personality fit, which by definition are subjective. Also, some experts might demonstrate reluctance to consider these criteria, and various opinions of several assessors are usual. These considerations render the process of supervisor selection a rather difficult real-life issue, which demonstrates the topicality and usability of the offered ff'-FFS-based MCDM framework. Here, we set the five criteria $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4$ and $\mathfrak{S}_5$ and five alternatives (supervisors) $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4$ and $\mathfrak{T}_5$, along with three decision-makers $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ for supervisors at research as well as academic institutions. A comprehensive explanation of the criteria along with decision makers are discussed below:

Field of Research ($\mathfrak{S}_1$): The key component of our Ph.D. careers is the research domain. Individuals generally think about doing research in their chosen field of study. However, on occasion students do not receive the real supervisor, who will carry out the study on their favorite subject. They are therefore experiencing a great deal of trouble with their Ph.D. studies. It requires a while for this to register. If he is appointed a supervisor of his decisions, he will be capable of comprehending the issue in their personal particular fashion. Therefore, when starting a research task, every one of the students must become aware of the professor's current study field and talk with our supervisor about it.

Publication Record ($\mathfrak{S}_2$): While selecting a supervisor, we may also consider their history of research publishing, including the quantity of published research papers along with the caliber of the journal within which their studies were published. Numerous scholars possess greater numbers of papers from national and international conferences than they do from journals; some have numerous research articles in not-of-high-quality journals, whereas others have limited research articles in prominent journals including SCI, SCOPUS, and others as well. We can consider each of these aspects while selecting a Ph.D. supervisor.

Academic expertise ($\mathfrak{S}_3$): In educational careers, supervisors can take on several roles, including assistant professor, associate professor, and professor. Assistant professors are often able to provide Ph.D. scholars with greater amounts of time compared to associate professors. Having more scholars compared to an assistant professor as well as an endless list of formal duties, an associate professor nonetheless finds time to mentor a student. Lastly, academics assign the smallest amounts of free time to their pupils while employing the most amounts of research scholars. Furthermore, they have a ton of job administration on their plate. They've got the greatest knowledge of research, though. The most senior professor's age is another important consideration. It is necessary to find a new supervisor if a professor departs while a student is pursuing their PhD. In this case, the subject matter can be problematic.

Role in administration ($\mathfrak{S}_4$): In most cases, a supervisor who serves as an associate professor or professor in an organization is required to be capable of carrying out specific administrative roles and duties. The Ph.D. scholars' direction is disrupted by these administrative duties because the mentors must dedicate their entire focus on completing them. This can negatively influence the guide-scholar connection and potentially have an impact on the research work in the future. However, given the above-indicated drawbacks, we may also analyze the benefits. Supervisors who hold crucial administrative positions inside the organization might be kinder when it comes to the accessibility of research equipment, facilitate greater research collaborations with other organizations, and offer employment chances within as well as outside the organization.

Lab equipment ($\mathfrak{S}_5$): Having a laboratory facility is necessary for Ph.D. study. We must inspect all required devices and amenities in the supervisor's lab while submitting our Ph.D. applications.

Table 2 shows the expert details, with their corresponding weights.

Table 2: Description of experts

Experts	Description	Weights
$\boldsymbol{\mathcal{D}_1}$	Student	0.31
$\boldsymbol{\mathcal{D}_2}$	Parents	0.33
$\boldsymbol{\mathcal{D}_3}$	Group friends	0.36

The hierarchical framework employed in the present investigation for choosing a suitable supervisor for higher studies is shown in Fig. 4.

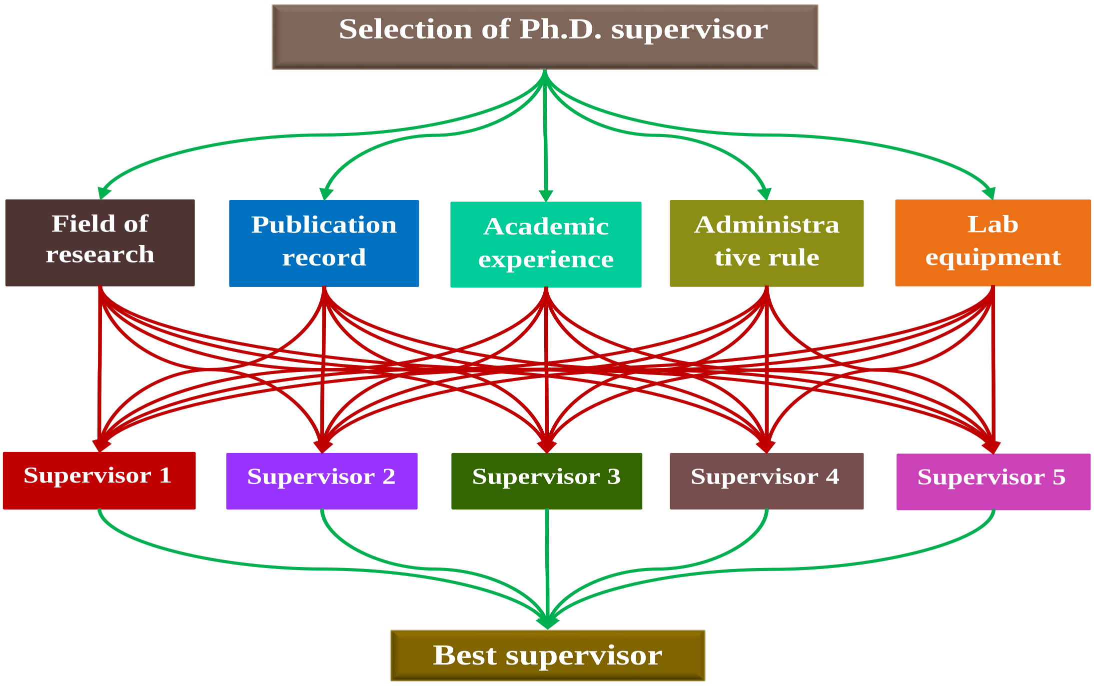

Figure 4: The hierarchical framework of the MCDM approach

7. Results and Discussion

In this section, we discuss the numerical outcomes of the proposed model in the process of selecting the best Ph.D. supervisor for higher studies of a student. Let us assume that the expert weights are 0.31, 0.33, and 0.36). Then the calculated results of the given information utilizing the proposed ff'-FF CODAS method are stepwise given as follows:

Step 1: Collect expert information in the form of ff'-FFNs. Tables 3-5 include all the information provided by three experts.

Table 3: 1st expert decision matrix

$\mathfrak{T}_i/\mathfrak{S}_j$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
$\mathfrak{T}_1$	(0.56,0.22)	(0.58,0.31)	(0.48, 0.29)	(0.51, 0.21)	(0.61, 0.36)
$\mathfrak{T}_2$	(0.49, 0.23)	(0.59,0.32)	(0.41, 0.31)	(0.52, 0.22)	(0.60, 0.37)
$\mathfrak{T}_3$	(0.39,0.24)	(0.58,0.33)	(0.51, 0.43)	(0.53, 0.23)	(0.64, 0.38)
$\mathfrak{T}_4$	(0.42,0.26)	(0.57, 0.27)	(0.53, 0.44)	(0.54, 0.24)	(0.59, 0.40)
$\mathfrak{T}_5$	(0.43,0.32)	(0.56, 0.23)	(0.52, 0.45)	(0.55, 0.25)	(0.50, 0.41)

Table 4: 2nd expert decision matrix

$\mathfrak{T}_i/\mathfrak{S}_j$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
$\mathfrak{T}_1$	(0.56, 0.12)	(0.42, 0.13)	(0.68, 0.17)	(0.62, 0.20)	(0.49, 0.14)
$\mathfrak{T}_2$	(0.57, 0.16)	(0.32, 0.14)	(0.67, 0.32)	(0.52, 0.24)	(0.65, 0.24)
$\mathfrak{T}_3$	(0.58, 0.19)	(0.33, 0.15)	(0.57, 0.33)	(0.47, 0.27)	(0.57, 0.21)
$\mathfrak{T}_4$	(0.59, 0.13)	(0.34, 0.16)	(0.45,0.34)	(0.53, 0.21)	(0.49, 0.19)
$\mathfrak{T}_5$	(0.63, 0.33)	(0.35, 0.17)	(0.36, 0.21)	(0.54, 0.28)	(0.52, 0.28)

Table 5: 3rd expert decision matrix

$\mathfrak{T}_i/\mathfrak{S}_j$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
$\mathfrak{T}_1$	(0.62, 0.19)	(0.41, 0.20)	(0.61, 0.19)	(0.59, 0.23)	(0.60, 0.27)
$\mathfrak{T}_2$	(0.57, 0.20)	(0.55, 0.21)	(0.67, 0.22)	(0.62, 0.32)	(0.43, 0.24)
$\mathfrak{T}_3$	(0.52, 0.15)	(0.59, 0.23)	(0.58, 0.24)	(0.53, 0.35)	(0.61, 0.18)
$\mathfrak{T}_4$	(0.51, 0.27)	(0.57, 0.26)	(0.52, 0.15)	(0.54, 0.29)	(0.53, 0.23)
$\mathfrak{T}_5$	(0.57, 0.16)	(0.48, 0.27)	(0.45, 0.14)	(0.48, 0.31)	(0.57, 0.27)

Step 2: Using the known expert weight information given in Table 1, aggregate the given expert information using ff'-FFWA operator (9). Table 6 shows the aggregated expert information, whereas Table 7 has the known criteria weight information.

Table 6: Aggregated matrix (using ff'-FFWA operator)

$\mathfrak{T}_i/\mathfrak{S}_j$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
$\mathfrak{T}_1$	(0.5835, 0.1708)	(0.4742, 0.1987)	(0.6024, 0.2088)	(0.5786, 0.2135)	(0.5674, 0.2377)
$\mathfrak{T}_2$	(0.5477, 0.1940)	(0.5027, 0.2093)	(0.6078, 0.2769)	(0.5600, 0.2591)	(0.5676, 0.2745)
$\mathfrak{T}_3$	(0.5073, 0.1876)	(0.5179, 0.2234)	(0.5570, 0.3194)	(0.5120, 0.2821)	(0.6080, 0.2407)
$\mathfrak{T}_4$	(0.5150, 0.2097)	(0.5079, 0.2241)	(0.5023, 0.2743)	(0.5376, 0.2458)	(0.5383, 0.2563)
$\mathfrak{T}_5$	(0.5554, 0.2519)	(0.4710, 0.2205)	(0.4476, 0.2298)	(0.5235, 0.2804)	(0.5338, 0.2633)

Table 7: Criteria weights

Criteria	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
weights	0.11	0.14	0.24	0.25	0.26

Step 5: Calculate the Euclidean distance (ED_i) and Hamming distance HD_i by utilizing Eq. (26) and Eq. (27). The numerical output is shown in Table 10.

Table 8: Weight normalized matrix ($\mathfrak{R}$).

$\mathfrak{T}_i/\mathfrak{S}_j$	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
$\mathfrak{T}_1$	0.0416	0.0333	0.0884	0.0845	0.0792
$\mathfrak{T}_2$	0.0352	0.0363	0.0756	0.0694	0.0708
$\mathfrak{T}_3$	0.0310	0.0368	0.0526	0.0512	0.0903
$\mathfrak{T}_4$	0.0299	0.0353	0.0484	0.0662	0.0667
$\mathfrak{T}_5$	0.0307	0.0303	0.0442	0.0547	0.0638

Step 6: Built the relative assessment matrix (RA) by using Eq. (28). Table 11 present the computed outcome of the relative assessment matrix.

Table 9: Negative ideal solution (NIS)

Criteria	$\mathfrak{S}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{S}_5$
$\boldsymbol{\mathcal{NS}}$	0.0299	0.0303	0.0442	0.0512	0.0638

Step 7: Calculate the assessment score values using Equation (31). Table 12 displays the outcome of the assessment score values, while table 13 shows the final ranking.

Table 10: Euclidean and Hamming distances

Alternatives	**Euclidean Distances (ED_i)**	**Hamming Distance (HD_i)**
$\mathfrak{T}_1$	0.0587	0.1077
$\mathfrak{T}_2$	0.0378	0.0679
$\mathfrak{T}_3$	0.0286	0.0426
$\mathfrak{T}_4$	0.0166	0.0271
$\mathfrak{T}_5$	0.0035	0.0042

Table 11: Relative Assessment matrix

Alternatives	$\mathfrak{T}_1$	$\mathfrak{T}_2$	$\mathfrak{T}_3$	$\mathfrak{T}_4$	$\mathfrak{T}_5$
$\mathfrak{T}_1$	0.0000	0.0210	0.0301	0.0422	0.0553
$\mathfrak{T}_2$	-0.0209	0.0000	0.0092	0.0212	0.0343
$\mathfrak{T}_3$	-0.0301	-0.0091	0.0000	0.0120	0.0251
$\mathfrak{T}_4$	-0.0420	-0.0212	-0.0120	0.0000	0.0131
$\mathfrak{T}_5$	-0.0550	-0.0342	-0.0250	-0.0130	0.0000

The graphical visualization of the outcomes of the proposed ff'-FF CODAS methodology is shown in Fig. 5.

Table 12: Assessment score values

Alternatives	$(\mathcal{AS}_i)$
$\mathfrak{T}_1$	0.1486
$\mathfrak{T}_2$	0.0438
$\mathfrak{T}_3$	-0.0021
$\mathfrak{T}_4$	-0.0621
$\mathfrak{T}_5$	-0.1272

Table 13: Final Ranking

Alternatives	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$

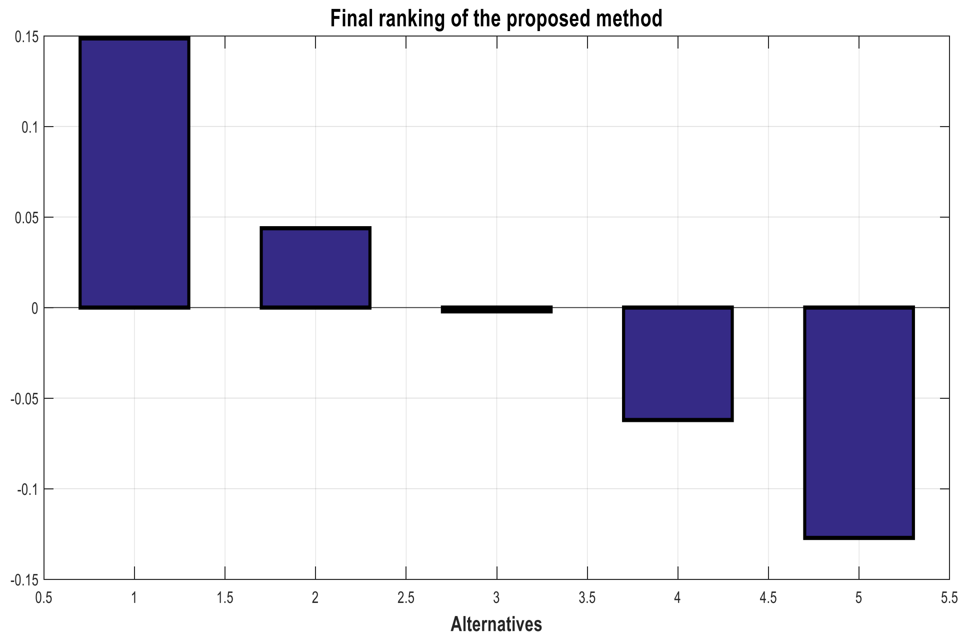

Figure 5: Graphical display of the final ranking via ff'- FF CODAS model

8. *Comparative Analysis*

In this section, we discuss the comparison of the proposed model with various existing aggregation operators to validate the stability and reliability of the proposed decision-making model. The proposed ff'-fractional fuzzy structure is the extension of the fractional fuzzy structure, which provides greater flexibility to the experts in dealing with hesitancy and uncertainty. It is therefore important to check how stable the proposed structure is. For this reason, we select the same expert-aggregated information with the same weights. We use the existing intuitionistic fuzzy weighted averaging IFWA [35], intuitionistic fuzzy weighted geometric IFWG [35], Pythagorean fuzzy weighted averaging PFWA [36], Pythagorean fuzzy weighted geometric PFWG [37], proposed ff'-fractional fuzzy weighted averaging (ff'-FFWA), proposed ff'-fractional fuzzy weighted geometric (ff'-FFWG) operators, WS approach [21], WP approach [22], and WASPAS approach [23]. A brief discussion of how various existing operators are used to rank the outcomes of alternatives is presented below.

(1) ***Comparing the proposed model with intuitionistic fuzzy aggregation operators:*** Inutility, we compare the proposed model with the IFWA and IFWG operators to show how successful it is. Since information is collected in the context of ff'-fractional fuzzy set, to perform this, we initially convert the ff'-fractional fuzzy information to IFS by applying the restriction ($f = f' = \frac{p}{q} \geq 1, p = q = 1$). Table 14 shows the IFWA along with IFWG score values for alternatives 0.3566, 0.3112, 0.2904, 0.2740, 0.2528, and 0.4079, 0.3656, 0.3516, 0.3344, 0.2027. Table 14 show that the optimal raking sequence for the alternative is $\mathfrak{T}_1$, which is identical to the proposed method.

(2) ***Comparing the proposed model with Pythagorean fuzzy aggregation operators:*** To facilitate contrast, we implement additional constraints ($f = f' = \frac{p}{q} \geq 1, p = 2q$) to ff'-fractional fuzzy set information to transform it towards PFS. The PFWA along with PFWG scores for alternatives are 0.2809, 0.2595, 0.2373, 0.2127, 0.1941 and 0.2567, 0.2242, 0.2079, 0.1881, 0.1637, respectively (refer to Table. 13). Table. 13 shows that the suitable alternative is still the same as the proposed method.

(3) ***Comparing the proposed model with the proposed aggregation operators:*** Considering the novel ff'-FFWA along with ff'-FFWG aggregation operators, the score values for alternatives are 0.3877, 0.3579, 0.3367, 0.3175, 0.2145 and 0.1828, 0.1042, 0.0.796, 0.0693 and 0.0298, respectively. Again, it is clear from Table 14 that the best alternative is $\mathfrak{T}_1$ which the same one as obtained in the suggested approach.

(4) ***Comparing the proposed model with existing MCDM models:*** We contrast the suggested technique to current WS, WP, and WASPAS approaches to show its effectiveness. For this purpose, we implement the WS approach [21], WP approach [22], and WASPAS approach [23] using expert information, while the raking is shown in Table 14, demonstrating that the alternative $\mathfrak{T}_1$ is same to our proposed method. As a result, the proposed model is extremely efficient in order to solve decision-making challenges in detail.

The complete visual explanation of the comparison of the proposed model with various existing frameworks is clearly depicted in Fig. 6.

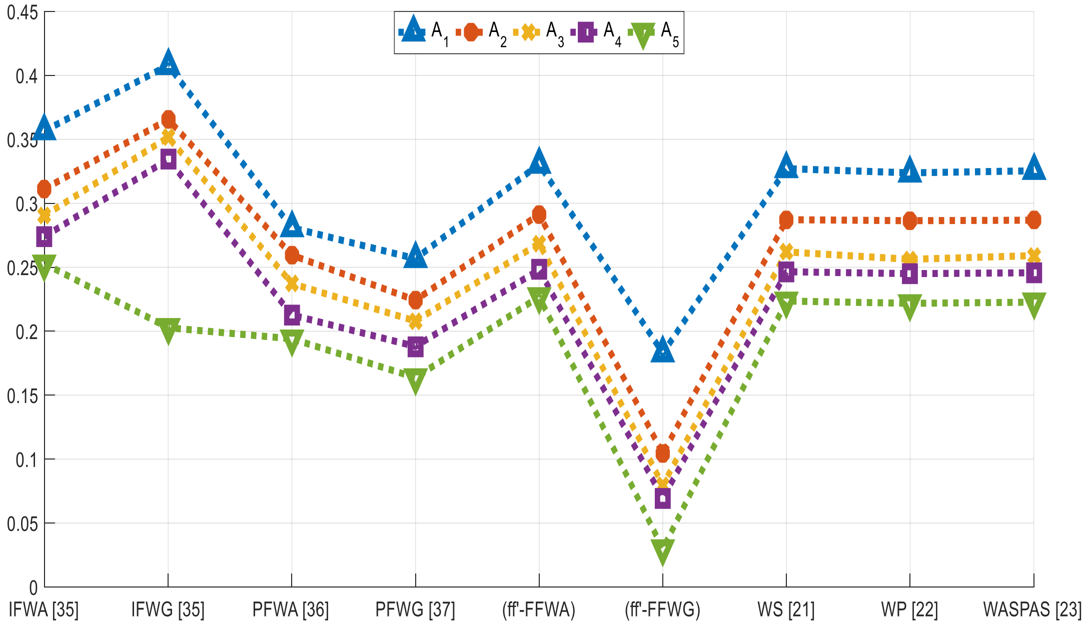

Figure 6: Graphical view of the comparison with existing frameworks.

Table 14: The results of the comparisons with existing operators and methods

Operators	**Score values**					**Ranking**
IFWA [35]	0.3566	0.3112	0.2904	0.2740	0.2528	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
IFWG[35]	0.4079	0.3656	0.3516	0.3344	0.2027	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
PFWA [36]	0.2809	0.2595	0.2373	0.2127	0.1941	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
PFWG [37]	0.2567	0.2242	0.2079	0.1881	0.1637	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
(ff'-FFWA)	0.3306	0.2914	0.2687	0.2481	0.2270	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
(ff'-FFWG)	0.1828	0.1042	0.0796	0.0693	0.0298	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
WS [21]	0.3271	0.2872	0.2620	0.2465	0.2236	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
WP [22]	0.3238	0.2864	0.2562	0.2450	0.2218	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
WASPAS [23]	0.3254	0.2868	0.2591	0.2457	0.2227	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$

9. *Sensitive Analysis*

In this section, we discuss the sensitivity analysis based on the parameter f and f'. The impact of parameters f as well as f' in analyzing the ranking outcomes is covered here. To create an improved depiction regarding the framework, we initially applied the proposed method to change the parameters f and f'. Selecting different values for parameters affects the decision-making outcomes that are displayed in Tables 15 as well as 16. Table 15 discusses the ranked arrangement of alternatives utilizing ff'-FF CODAS approach by factoring in the interval (1, 4), respectively. From table 16, we observe clearly that by using the ff'-CODAS approach, the outcomes of each alternative decrease as f and f' increase from interval $(1,2)$ to $(3,4)$ but $\mathfrak{T}_5$ increases from -0.113 to -0.067. However, the final ranking is still the same. Hence, the best Ph.D. supervisor is $\mathfrak{T}_1$. The graphical illustration is shown in the Fig. 7. The outcome of the alternatives ranking is then examined when the parameters f along with f' rise from $f, f' \in (4,5)$ to $f, f' \in (16,17)$. Table 15 makes it evident that the overall ranking of alternatives shifts from $\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$ to $\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$, as a result of the rise in f and f'. In this case, $\mathfrak{T}_2$ is the most suitable alternative. Additionally, when the values of f and f' rise, the results of each alternative go down. The ranking is unchanged for $f, f' \geq 17$, i.e., $\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$, and the alternative amount decreases. Table 15 shows the quantitate outcomes of the suggested method in the respective intervals. Overall, the discussion illustrates that the proposed technique is capable of providing reliable and clear rankings under various parameter settings and accurately capturing the dynamic influence of f and f prime on alternative quality. This highlights its application to real-world decision-making problems where parameter uncertainty and sensitivity are critical.

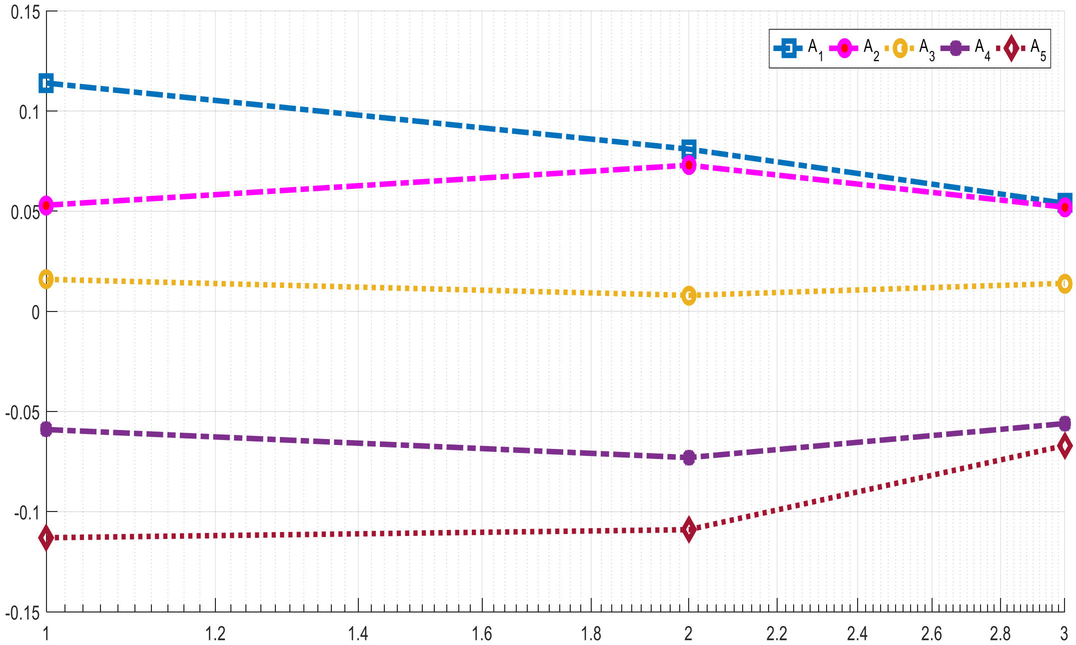

Figure 7: Sensitivity Analysis based on f and f' for the proposed ff'-FF CODAS approach

Table 15: Sensitive Analysis based on the parameters f and f'

Values of f and f'	Outcomes of ff' − FF CODAS approach	Ranking
$f, f' \in (1, 2)$	0.114 0.053 0.016 -0.059 -0.113	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (2, 3)$	0.081 0.073 0.008 -0.073 -0.109	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (3, 4)$	0.054 0.052 0.014 -0.056 -0.067	$\mathfrak{T}_1 \succ \mathfrak{T}_2 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$

Table 16: Sensitivity Analysis based on f and f'

f and f'	Outcomes of ff'-CODAS approach	Ranking
$f, f' \in (4, 5)$	0.035 0.046 0.011 -0.042 -0.049	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (5, 6)$	0.023 0.035 0.007 -0.031 -0.034	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (6, 7)$	0.015 0.026 0.004 -0.022 -0.023	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (7, 8)$	0.010 0.018 0.002 -0.015 -0.016	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (8, 9)$	0.007 0.013 0.001 -0.010 -0.010	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (9, 10)$	0.005 0.009 0.0002 -0.00 71 -0.007	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$
$f, f' \in (16, 17)$	0.0002 0.00062 -0.0001 -0.0004 -0.0004	$\mathfrak{T}_2 \succ \mathfrak{T}_1 \succ \mathfrak{T}_3 \succ \mathfrak{T}_4 \succ \mathfrak{T}_5$

The graphical explanation of Table 16 is shown in the Fig 8.
It follows that $\mathfrak{T}_2$ is the most suitable alternative (Ph.D. supervisor), next to $\mathfrak{T}_1$ as well as $\mathfrak{T}_3$. Even though alternative $\mathfrak{T}_4$ is negative, it only makes an extremely little impact on the proposed method. However, the alternative $\mathfrak{T}_5$ is completely cost-prohibitive and has the opposite effect on this model. Such comparisons are made feasible by the adaptable parameters of the ff'-FFWA operator.

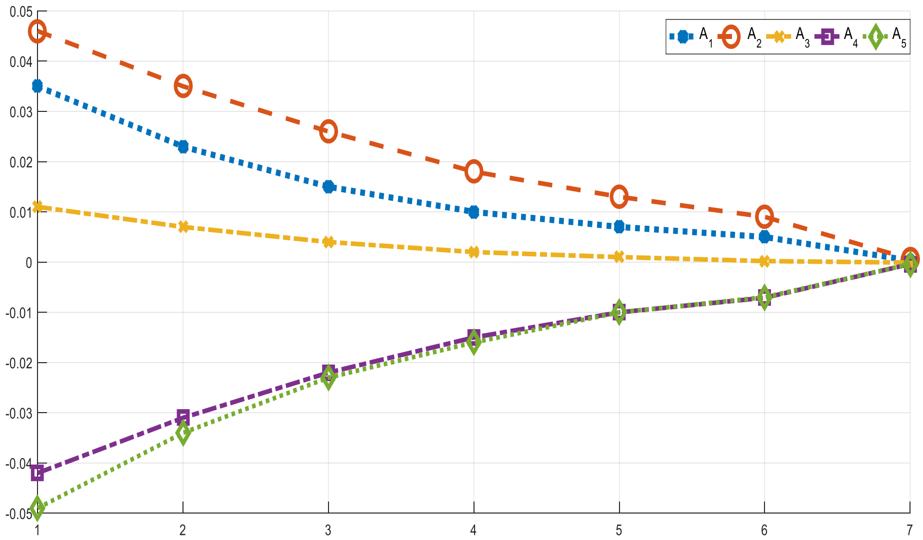

Figure 8: Sensitive Analysis based on parameters f and f' for ff'-FF CODAS approach

10. Conclusion

The selection of best supervisor for higher studies is generally a key decision for students starting the research in a particular filed. Choosing a best supervisor in an academic career can be a complex task for students. This paper proposes a new structure called ff'-fractional fuzzy sets (ff'-FFSs). This structure is a continuous structure and a generalization of all fuzzy sets extension, which contain MD and NMD. It provides and additional for the expert to convey their assessment information more accurately. Various aggregating operators, including arithmetic mean, geometric mean, weighted averaging, and weighted geometric operators based on ff'-FFNs, have been designed and created to combine expert information. Next, we proposed a novel decision-making method called ff'-FF CODAS method for addressing the MCDM challenge using ff'-FFSs. The proposed method is a well-known MCDM method which is famous for hamming and Euclidean distance measures. A comprehensive explanation of the ff'-FF CODAS method is delivered in the context of ff'-FFSs. The proposed approach's scheme works well for ff'-FF data. The proposed method stepwise algorithm is displayed in the flow chart. Following that, a mathematical description of the suggested approach is provided, and various operators along with MCDM approaches confirmed the ranking conclusions. Supervisor $\mathfrak{T}_1$ is the best choice when ranking alternatives using the ff'-FF CODAS approach, followed by Supervisor $\mathfrak{T}_2$ up to so on. Additionally, to verify the equilibrium and improve the assessment of the proposed method, sensitive analysis and comparative analysis are also conducted. Overall, the proposed ff'-FF CODAS method is a helpful tool for real-world decision-making problems.

10.1. Future Work

The proposed work can be extended in future studies by combining it with neural network-based solutions, including ff'-fractional fuzzy neural networks, to enhance predictive performance and find more complex relationships in uncertain data. The framework shall be tested in case studies those are related to renewable energy, economic policy, supply chain, and healthcare applications. Also, we will try to develop the Hamacher, Einstein, Frank, Yager, and Aczel-Alsina aggregation operators and compare them to determine the effect of these aggregation operators on decision-making outcomes. The robustness, reliability, and practical applicability of the approach shall be assured by further sensitivity and comparative analyses.

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***Human Participants and/or Animals**:* This research did not involve any human participants or animals.

Ethics approval: This paper does not contain any studies with animals performed by any of the authors.

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