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Development of Non-linear Diophantine Linguistic Term Sets and Their Application in Decision-Making Problems

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ABSTRACT

In this study, we introduce a new notion called non-linear Diophantine linguistic term sets, which is a hybrid of the non-linear Diophantine fuzzy set and linguistic term sets. Initially, we will go over the fundamental idea of fuzzy sets and its generalization. Discuss both linear and nonlinear Diophantine fuzzy sets. Following that, we describe the non-linear Diophantine linguistic term sets, as well as their score and accuracy functions. Next, we define the operational rules and aggregation operator for the proposed notion using the Hamacher t-norm and t-conorm. Furthermore, we modify the TOPSIS approach for non-linear Diophantine linguistic term sets using the Hamacher aggregation operator. Next, we apply the proposed approach to the selection of mathematical thinking tools, gathering reliable guidance from three experts. To aggregate these experts' data, we use the Hamacher aggregation operator, followed by the modified TOPSIS approach for the final data. According to the presented framework, Abstraction is the most effective mathematical tool. Finally, we compare the proposed model to existing approaches to ensure its accuracy and applicability. Comparative study shows that the proposed framework appears to be accurate and flexible to real-world decision-making challenges.

Keywords

Non-linear Diophantine linguistic term sets; TOPSIS method; Hamacher aggregation operator; thinking tools

1. *Introduction*

Multi criteria decision making (MCDM) [1-3], a subfield of operations research, deal with identifying the optimal answer. When making decisions, viable solutions are explicitly weighed against conflicting criteria. MCDM is a crucial part of contemporary MCDM frameworks. DMs usually have to provide information on the evaluation across several sorts of qualities since the actual world is so complex, attributes are unclear, and individual judgments are subjective.

The fuzzy set (FS) concept with membership degree (MD), first proposed by Zadeh [4] in 1965, is a useful technique for managing unclear and confusing data in everyday life. In addition to FSs, he introduced the important concept of linguistic variables [5] (LVs). We can solve multi criteria decision making (MCDM) issues using a variety of mathematical techniques and translate conversations into mathematical expressions using LVs. In 1986, Atanassov [6] developed the idea of intuitionistic fuzzy sets (IFS) by building based on FSs. In IFS, the total of the MD and nonmembership degree (NMD) must equal one or less. In some instances, the DMs may allocate some feature to the MDs and NMDs such that their combined totals exceed one, when the total of their squares is one or less. In order to address this issue, Yager [7] developed the Pythagorean fuzzy set (PFS), which is a type of IFS. An MD and NMD squared combined that is less than or equal to one is called PFS. While the development of PFS has solved many issues, there are still a number of cases it is unable to solve. Take for example a decision maker with a MD of 0.9 and NMD of 0.6; such a case would exhibit IFS and PFS would be powerless to help. To resolve this issue, T. Senapati and Yager [8] proposed the Fermatean fuzzy set (FFS) with the stipulation that $0 \leq MD^3 + NMD^{\,3} \leq 1$. Although FFS has resolved many cases, there are still many instances in which it has been unable to overcome challenges. For instance, IFS, PFS and FFS cannot resolve the issue if a decision-maker is presented with an MD of 0.9 and an NMD of 0.8. With the caveat that the total of the q-power of MD and NMD must be between zero and one, Yager [9] proposed the idea of a q-rung orthopair fuzzy set (q-ROFS) to address these types of scenarios. In essence, q-ROFS integrated the ideas of FS, IFS, PFS and FFS. Numerous researchers from a range of fields have employed q-ROFSs; most recently, they formulated and resolved the

MCDM issues utilizing tangential-based operational principles and q-ROFS aggregation operators [10-13]. Figure 1 show the comparison of these concepts.

The Linear Diophantine Fuzzy Set (LDFS), created by Riaz and Hashmi [14], addresses the drawbacks of existing methods by combining MG and NMG with reference parameters (RPs). Because RPs is included, the LDFS model is more effective and accurate than other fuzzy models. The range of 0 to 1 in LDFSs represents the total of RPs with product to MG and NMG, respectively. However, LDFS has limited itself to reaching its RPs aim since in some real-life issues, the total number of RPs that an option meets the requirements attained by DM is frequently larger than one. Almagrabi et al. [15] presented the notion of a q-rung linear Diophantine fuzzy set (q-RLDFS). The q^{th} power was added to RPs that included the available space of the current MG and NMG structure associated with RPs. The MG, NMG, and RPs all contribute to the idea of q-RLDFS; the total of the q^{th} power to RPs pertaining to MG and NMG falls between zero and one. In the case of LDFS, the MADM issue is limited since the sum of the RPs provided by DM may be more than one. This requirement can be addressed by the q-RLDFS idea, which eliminates the LDFS incompatibility. The set of feasible Diophantine grows as the q^{th} rises, and more Diophantine satisfy the border restriction. The core benefit of the q-RLDF way is that it reflects the qth power of RPs, suggesting that it may be charity to real-world DM issues. The q-RLDFS, a unique LDF extension, and looked at its salient features. The idea of q-RLDFSs is a original way to characterize uncertainty in DM. As it syndicates the qth power of RFs with MG and NMG, which were also introduced, the q-RLDFS is more adaptable and reliable than present q-ROFSs and LDFSs and it is mentioned to as the Non-LDFSs (N-LDFS). Many researchers utilized these concepts to solve MCDM challenges [16-20].

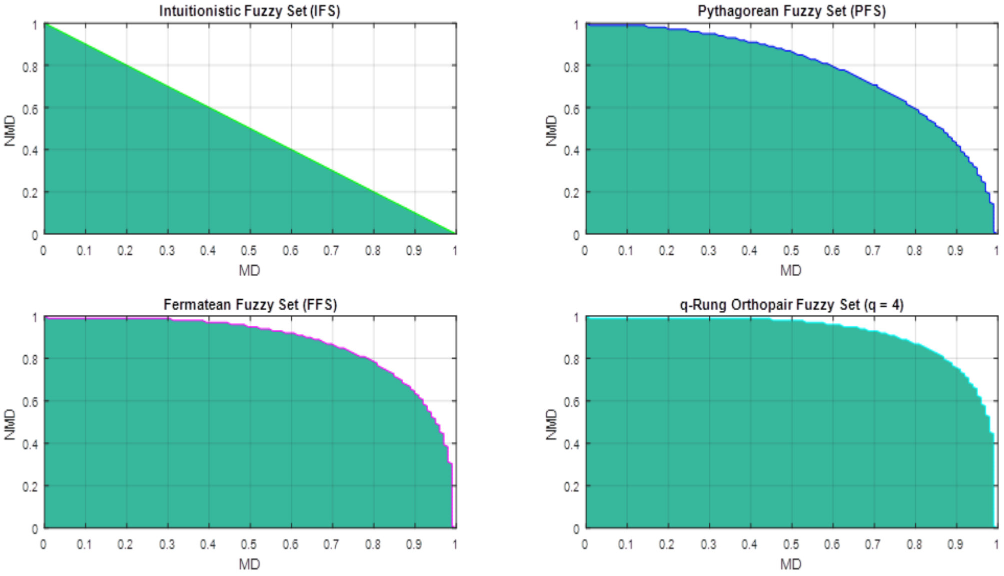

Figure 1: graphical representation of IFS, PFS, FFS and q-ROFS.

In this work, we present the new idea of N-LD-Linguistic Term Sets (N-LDLTS), which combines N-LDFS with linguistic term sets. We adjust the TOPSIS method for the NLDLTS. The primary contributions of this work are as follows:

i. The novel idea of N-LDLTS.
ii. The establishment of the operational rules and aggregation operator for N-LDLTS.
iii. The description of the accuracy and score function for the N-LDLTS.
iv. Modification of the TOPSIS approach under the Nonlinear Diophantine linguistic word sets.
v. The analyses of the mathematical thinking tools based on the given model and choose the best one.
vi. Assessment of the model's accuracy and reliability by comparing it to the existing model.

2. Foundational Ideas

In this section, present LDFS and N-LDFS.

Definition 2.1: [14] Let$\aleph$ be universal set$\aleph$. Then, LDFS $\mathcal{D}$ on $\aleph$ is given by:

$$\mathcal{D} = \{\langle \mathfrak{C}, \mathfrak{m}_{\mathcal{D}}(\mathfrak{C}), \mathfrak{n}_{\mathcal{D}}(\mathfrak{C}) \rangle, \langle \alpha, \beta \rangle | \mathfrak{C} \in \aleph\}, \quad (2.1)$$

Where $\mathfrak{m}_{\mathcal{D}}(\mathfrak{C}): \aleph \to [0,1]$, and $\mathfrak{n}_{\mathcal{D}}(\mathfrak{C}): \aleph \to [0,1]$denotes the functions for MD and NMD. $\langle \alpha, \beta \rangle$ Represent reference parameters which belong to [0,1] with the following condition:

$$0 \leq \alpha \mathfrak{m}_{\mathcal{Q}}(\mathfrak{C}) + \beta \mathfrak{n}_{\mathcal{Q}}(\mathfrak{C}) \leq 1 \text{ With } 0 \leq \alpha + \beta \leq 1$$

Definition 2.2: [15] Consider the universal set$\aleph$. Then, N-LDFS$\mathfrak{P}$ on $\aleph$ is described as follows:

$$\mathfrak{P} = \left\{\langle \mathfrak{C}, \mathfrak{m}_{\mathfrak{P}}(\mathfrak{C}), \mathfrak{n}_{\mathfrak{P}}(\mathfrak{C}) \rangle, \langle \alpha, \beta \rangle | \mathfrak{C} \in \aleph\right\}, \quad (2.2)$$

Where $\mathfrak{m}_{\mathfrak{P}}(\mathfrak{C}): \aleph \to [0,1]$, and $\mathfrak{n}_{\mathfrak{P}}(\mathfrak{C}): \aleph \to [0,1]$denotes the MD and NMD respectively, $\langle \alpha, \beta \rangle$ representreference parameters belong to [0,1] with the condition:

i. $0 \leq \alpha^{r} \mathfrak{m}_{\mathfrak{P}}(\mathfrak{C}) + \beta^{r} \mathfrak{n}_{\mathfrak{P}}(\mathfrak{C}) \leq 1$ with $r \geq 1$
ii. $0 \leq \alpha^{r} + \beta^{r} \leq 1$

Definition 2.3: Consider the universal set$\aleph$. Then, N-LDLTS$\mathfrak{J}$ on $\aleph$ is described as follows:

$$\mathfrak{I}=\left\{\langle \mathfrak{C}, \mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I})}}\rangle, \langle \alpha, \beta\rangle | \mathfrak{C} \in \aleph\right\}, \tag{2.3}$$

Where $\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I})}}: \aleph \rightarrow [0, \mathscr{b}]$, and $\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I})}}: \aleph \rightarrow [0, \mathscr{b}]$denotes the MD and NMD respectively, $\langle \alpha, \beta\rangle$ representreference parameters belong to [0, $\mathscr{b}$] with the condition:

i. $0 \leq \frac{\alpha^{\mathscr{r}} \mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I})}}}{\mathscr{b}^2}+\frac{\beta^{\mathscr{r}} \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I})}}}{\mathscr{b}^2} \leq 1$ with $\mathscr{r} \geq 1$

ii. $0 \leq \frac{\alpha^{\mathscr{r}}}{\mathscr{b}}+\frac{\beta^{\mathscr{r}}}{\mathscr{b}} \leq 1$,

where,$\mathcal{S}_{\Omega} | \Omega=0,1,2,3, \ldots, \mathscr{b}$ represent the LTs with lower and higher limits are denoted by $\mathcal{S}_0$ and $\mathcal{S}_{\mathscr{b}}$ of the LTs respectively.

Definition2.4: If $\mathfrak{I}_i$ is N-LDFS, then the score function of N-LDLTS are described as follows:

$$Sf(\mathfrak{I}_i)=\frac{1}{2}\left[\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)+\left(\frac{{\alpha_i}^{\mathscr{r}}}{\mathscr{b}}-\frac{{\beta_i}^{\mathscr{r}}}{\mathscr{b}}\right)\right] \tag{2.4}$$

Definition2.5: If $\mathfrak{I}_i$ is N-LDFS, then the accuracy function of N-LDLTS are described as follows:

$$Af(\mathfrak{I}_i)=\frac{1}{2}\left[\frac{\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}+\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)}{2}+\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}+\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right] \tag{2.5}$$

Definition 2.6: If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are two N-LDLTSs, then the two N-LDLTSs compered based on score and accuracy functions:

i. If $Sf(\mathfrak{I}_1)<Sf(\mathfrak{I}_2)$then $\mathfrak{I}_1<\mathfrak{I}_2$

ii. If $Sf(\mathfrak{I}_1)>Sf(\mathfrak{I}_2)$then $\mathfrak{I}_1>\mathfrak{I}_2$

iii. If $Sf(\mathfrak{I}_1)=Sf(\mathfrak{I}_2)$ then,

a. $Af(\mathfrak{I}_1)<Af(\mathfrak{I}_2)$then $\mathfrak{I}_1<\mathfrak{I}_2$

b. $Af(\mathfrak{I}_1)>Af(\mathfrak{I}_2)$then $\mathfrak{I}_1>\mathfrak{I}_2$

c. $Af(\mathfrak{I}_1)=Af(\mathfrak{I}_2)$then $\mathfrak{I}_1 \cong \mathfrak{I}_2$

Definition 2.7*:* Let $\mathbb{T}$ and $\mathbb{S}$ be Hamacher norms [21] and can be define as:

$$\mathbb{T}(x, y)=\frac{xy}{\varphi+(1-\varphi)(x+y-xy)}, \varphi \geq 0$$

$$\mathbb{S}(x, y)=\frac{(x+y-xy)-(1-\varphi)xy}{1-(1-\varphi)xy}, \varphi \geq 0$$

Definition 2.8: If $\mathfrak{I}_1=\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right), \\ (\alpha_1, \beta_1)\end{matrix}\right\}$ and $\mathfrak{I}_2=\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}\right), \\ (\alpha_2, \beta_2)\end{matrix}\right\}$ are two N-LDLTSs, then the essential properties of N-LDLTSs are described as follows:

i.

$$\mathfrak{I}_1 \oplus_H \mathfrak{I}_2=\left\{\begin{matrix}\left(\begin{matrix}\mathscr{b}\left(\frac{\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}+\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}-\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)-(1-\varphi)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)}{1-(1-\varphi)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)}\right), \\ \mathscr{b}\left(\frac{\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}}{\varphi+(1-\varphi)\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}+\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}}\right)\end{matrix}\right), \\ \left(\begin{matrix}\mathscr{b}\left(\sqrt[\mathscr{r}]{\frac{\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{\mathscr{r}}+\left(\frac{\alpha_2}{\mathscr{b}}\right)^{\mathscr{r}}-\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{\mathscr{r}}\left(\frac{\alpha_2}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)-(1-\varphi)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{\mathscr{r}}\left(\frac{\alpha_2}{\mathscr{b}}\right)^{\mathscr{r}}\right)}{1-(1-\varphi)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{\mathscr{r}}\left(\frac{\alpha_2}{\mathscr{b}}\right)^{\mathscr{r}}\right)}}\right), \\ \mathscr{b}\left(\frac{\left(\frac{\beta_1}{\mathscr{b}}\right)^{\mathscr{r}}\left(\frac{\beta_2}{\mathscr{b}}\right)^{\mathscr{r}}}{\sqrt[\mathscr{r}]{\varphi+(1-\varphi)\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{\mathscr{r}}+\left(\frac{\beta_2}{\mathscr{b}}\right)^{\mathscr{r}}\right)-\left(\frac{\beta_1}{\mathscr{b}}\right)^{\mathscr{r}}\left(\frac{\beta_2}{\mathscr{b}}\right)^{\mathscr{r}}}}\right)\end{matrix}\right)\end{matrix}\right\}$$

ii.

$$\mathfrak{I}_1 \otimes_H \mathfrak{I}_2 = \left\{\begin{array}{c}
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}}{\varphi+(1-\varphi)\left(\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}+\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)-\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}}\right), \\
\mathscr{b}\left(\dfrac{\left(\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}+\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}-\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)-(1-\varphi)\left(\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)}{1-(1-\varphi)\left(\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\frac{\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}}{\sqrt[r]{\varphi+(1-\varphi)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}+\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}\right)-\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}}}\right), \\
\mathscr{b}\left(\sqrt[r]{\dfrac{\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}+\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}-\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}\right)\right)-(1-\varphi)\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}\right)}{1-(1-\varphi)\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}\right)}}\right)
\end{array}\right)
\end{array}\right\}$$

iii.

$$\zeta \odot \mathfrak{I}_1 = \left\{\begin{array}{c}
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\left(1+(\varphi-1)\left(\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\zeta}-\left(1-\left(\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\zeta}}{\left(1+(\varphi-1)\left(\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\zeta}+(\varphi-1)\left(1-\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)^{\zeta}}\right), \\
\mathscr{b}\left(\dfrac{\varphi\left(\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)^{\zeta}}{\left(1+(\varphi-1)\left(\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)\right)^{\zeta}-(\varphi-1)\left(1-\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)^{\zeta}}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\mathscr{b}\left(\sqrt[r]{\dfrac{\left(1+(\varphi-1)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)\right)^{\zeta}-\left(1-\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)\right)^{\zeta}}{\left(1+(\varphi-1)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)\right)^{\zeta}+(\varphi-1)\left(1-\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}}\right), \\
\mathscr{b}\left(\dfrac{\varphi\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}{\sqrt[r]{\left(1+(\varphi-1)\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)\right)^{\zeta}-(\varphi-1)\left(1-\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}}\right)
\end{array}\right)
\end{array}\right\}$$

iv.

$$\mathfrak{I}_1{}^{\zeta} = \left\{\begin{array}{c}
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\varphi\left(\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)^{\zeta}}{\left(1+(\varphi-1)\left(\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\zeta}-(\varphi-1)\left(1-\frac{\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)^{\zeta}}\right), \\
\mathscr{b}\left(\dfrac{\left(1+(\varphi-1)\left(\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)\right)^{\zeta}-\left(1-\left(\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)\right)^{\zeta}}{\left(1+(\varphi-1)\left(\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)\right)^{\zeta}+(\varphi-1)\left(1-\mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}\right)^{\zeta}}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\varphi\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}{\sqrt[r]{\left(1+(\varphi-1)\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}-(\varphi-1)\left(1-\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}}\right) \\
\mathscr{b}\left(\sqrt[r]{\dfrac{\left(1+(\varphi-1)\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)\right)^{\zeta}-\left(1-\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}{\left(1+(\varphi-1)\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}+(\varphi-1)\left(1-\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)^{\zeta}}}\right),
\end{array}\right)
\end{array}\right\}$$

Definition2.8: The collection of N-LDLTSs is represented by $\mathfrak{I}_i = \left\{\begin{array}{c}\left(\mathscr{s}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}, \mathscr{s}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}\right), \\ (\alpha_i, \beta_i)\end{array}\right\}$ $(i = 1,2,3,\dots,n)$, and $\Psi = (\Psi_1, \Psi_2, \Psi_3, \dots, \Psi_n)^{\mathfrak{z}}$ is a weight vector with $\Psi_i > 0$and $\sum_{i=1}^{n} \Psi_i = 1$. Then, N-LHDLWAA operator will be mapped by the N-LDLHWAA: $\mathfrak{I}_i{}^{n} \rightarrow \mathfrak{I}_i$.

N-LDLHWAA$(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \dots, \mathfrak{I}_n) = \oplus_{\mathrm{i}=1}^{\mathrm{n}} \Psi_i \mathfrak{I}_i$

$$
= \left\{ \begin{array}{c}
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}\left(1-\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} + \prod_{i=1}^{n}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right), \\
\mathscr{b}\left(\dfrac{\prod_{i=1}^{n}\varphi\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)^{\Psi_i}}{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right)
\end{array} \right), \\
\left(\begin{array}{c}
\mathscr{b}\left(\sqrt[r]{\dfrac{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}\left(1-\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i}}{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} + \prod_{i=1}^{n}(\varphi-1)\left(1-\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}} \right) \\
\mathscr{b}\left(\dfrac{\prod_{i=1}^{n}\varphi\left(\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}{\sqrt[r]{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}(\varphi-1)\left(1-\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}} \right),
\end{array} \right)
\end{array} \right\}
$$

Theorem 4.1: The collection of N-LDLTSs is represented by $\mathfrak{J}_i = \left\{ \begin{array}{c} \left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}\right), \\ (\alpha_i, \beta_i) \end{array} \right\}$, $(i = 1,2,3, \dots, n)$and $\Psi = (\Psi_1, \Psi_2, \Psi_3, \dots, \Psi_n)^{\mathfrak{z}}$ is a weight vector with $\Psi_i > 0$and $\sum_{i=1}^{n} \Psi_i = 1$. Then, non-linear Diophantine linguistic Hamacher weighted averaging aggregation (N-LDLHWAA) operator will be mapped by the N-LDLHWAA: ${\mathfrak{J}_i}^{n} \rightarrow \mathfrak{J}_i$. Then, the aggregated values of the N-LDLHWAA operator are again N-LDLTS.

$$\text{N-LDLHWAA}(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, .., \mathfrak{J}_n) = \oplus_{\mathrm{i}=1}^{\mathrm{n}} \Psi_i \mathfrak{J}_i$$

$$
= \left\{ \begin{array}{c}
\left(\begin{array}{c}
\mathscr{b}\left(\dfrac{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}\left(1-\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} + \prod_{i=1}^{n}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right), \\
\mathscr{b}\left(\dfrac{\prod_{i=1}^{n}\varphi\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)^{\Psi_i}}{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right)
\end{array} \right), \\
\left(\begin{array}{c}
\mathscr{b}\left(\sqrt[r]{\dfrac{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}\left(1-\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i}}{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} + \prod_{i=1}^{n}(\varphi-1)\left(1-\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}} \right) \\
\mathscr{b}\left(\dfrac{\prod_{i=1}^{n}\varphi\left(\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}{\sqrt[r]{\prod_{i=1}^{n}\left(1+(\varphi-1)\left(\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} - \prod_{i=1}^{n}(\varphi-1)\left(1-\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}} \right),
\end{array} \right)
\end{array} \right\}
$$

Proof: For this, we apply the mathematical induction approach.

For $n = 2$, we have

$$\text{N-LDLHWAA}(\mathfrak{J}_1, \mathfrak{J}_2) = \oplus_{\mathrm{i}=1}^{2} \Psi_i \mathfrak{J}_i$$

$$=\left\{\begin{array}{c}\left(\mathscr{b}\left(\frac{\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\Psi_1}-\left(1-\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\Psi_1}}{\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\Psi_1}+\ (\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)^{\Psi_1}}\right),\ \mathscr{b}\left(\frac{\varphi\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)^{\Psi_1}}{\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)\right)^{\Psi_1}-\ (\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_1)}}}{\mathscr{b}}\right)^{\Psi_1}}\right)\right),\\ \left(\mathscr{b}\left(\sqrt[r]{\frac{\left(1+(\varphi-1)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_1}-\left(1-\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_1}}{\left(1+(\varphi-1)\left(\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_1}+\ (\varphi-1)\left(1-\left(\frac{\alpha_1}{\mathscr{b}}\right)^{r}\right)^{\Psi_1}}}\right),\ \mathscr{b}\left(\frac{\varphi\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)^{\Psi_1}}{\sqrt[r]{\left(1+(\varphi-1)\left(\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_1}-\ (\varphi-1)\left(1-\left(\frac{\beta_1}{\mathscr{b}}\right)^{r}\right)^{\Psi_1}}}\right),\right)\end{array}\right\}$$

$$\oplus_H\left\{\begin{array}{c}\left(\mathscr{b}\left(\frac{\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)\right)^{\Psi_2}-\left(1-\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)\right)^{\Psi_2}}{\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)\right)^{\Psi_2}+\ (\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)^{\Psi_2}}\right),\ \mathscr{b}\left(\frac{\varphi\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)^{\Psi_2}}{\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)\right)^{\Psi_2}-\ (\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_2)}}}{\mathscr{b}}\right)^{\Psi_2}}\right)\right),\\ \left(\mathscr{b}\left(\sqrt[r]{\frac{\left(1+(\varphi-1)\left(\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_2}-\left(1-\left(\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_2}}{\left(1+(\varphi-1)\left(\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_2}+\ (\varphi-1)\left(1-\left(\frac{\alpha_2}{\mathscr{b}}\right)^{r}\right)^{\Psi_2}}}\right),\ \mathscr{b}\left(\frac{\varphi\left(\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}\right)^{\Psi_2}}{\sqrt[r]{\left(1+(\varphi-1)\left(\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_2}-\ (\varphi-1)\left(1-\left(\frac{\beta_2}{\mathscr{b}}\right)^{r}\right)^{\Psi_2}}}\right),\right)\end{array}\right\}$$

$$= \left\{ \begin{matrix} \left(\begin{matrix} \mathscr{b}\left(\frac{\prod_{i=1}^{2}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}-\prod_{i=1}^{2}\left(1-\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{2}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}+\prod_{i=1}^{2}(\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)^{\Psi_i}}\right), \\ \mathscr{b}\left(\frac{\prod_{i=1}^{2}\varphi\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)^{\Psi_i}}{\prod_{i=1}^{2}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}-\prod_{i=1}^{2}(\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)^{\Psi_i}}\right) \end{matrix} \right), \\ \left(\begin{matrix} \mathscr{b}\left(\sqrt[\mathscr{r}]{\frac{\prod_{i=1}^{2}\left(1+(\varphi-1)\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}-\prod_{i=1}^{2}\left(1-\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{2}\left(1+(\varphi-1)\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}+\prod_{i=1}^{2}(\varphi-1)\left(1-\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}}\right) \\ \mathscr{b}\left(\frac{\prod_{i=1}^{2}\varphi\left(\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}{\sqrt[\mathscr{r}]{\prod_{i=1}^{2}\left(1+(\varphi-1)\left(\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}-\prod_{i=1}^{2}(\varphi-1)\left(1-\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}}\right), \end{matrix} \right) \end{matrix} \right\}$$

Thus, for n=2 is proved.

Suppose for $n = k$ is proved

$$\text{N-LDLHWAA}(\Im_1, \Im_2, \Im_3, .., \Im_k) = \oplus_{\mathrm{i}=1}^{\mathrm{k}} \Psi_i \Im_i$$

$$= \left\{ \begin{matrix} \left(\begin{matrix} \mathscr{b}\left(\frac{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}-\prod_{i=1}^{k}\left(1-\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}+\prod_{i=1}^{k}(\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)^{\Psi_i}}\right), \\ \mathscr{b}\left(\frac{\prod_{i=1}^{k}\varphi\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)^{\Psi_i}}{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}-\prod_{i=1}^{k}(\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\Im_i)}}}{\mathscr{b}}\right)^{\Psi_i}}\right) \end{matrix} \right), \\ \left(\begin{matrix} \mathscr{b}\left(\sqrt[\mathscr{r}]{\frac{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}-\prod_{i=1}^{k}\left(1-\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}+\prod_{i=1}^{k}(\varphi-1)\left(1-\left(\frac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}}\right) \\ \mathscr{b}\left(\frac{\prod_{i=1}^{k}\varphi\left(\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}{\sqrt[\mathscr{r}]{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}-\prod_{i=1}^{k}(\varphi-1)\left(1-\left(\frac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}}\right), \end{matrix} \right) \end{matrix} \right\}$$

Now we check for$n = k + 1$, we have

$$\text{N-LDLHWAA}(\Im_1, \Im_2, \Im_3, .., \Im_k, \Im_{k+1}) = \oplus_{\mathrm{i}=1}^{\mathrm{k}+1} \Psi_i \Im_i$$

$$
= \left\{ \begin{array}{l}
\left(\mathscr{b}\left(\dfrac{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{k}\left(1-\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} + \prod_{i=1}^{k}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right), \right. \\
\left. \mathscr{b}\left(\dfrac{\prod_{i=1}^{k}\varphi\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)^{\Psi_i}}{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{k}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right) \right), \\
\left(\mathscr{b}\left(\sqrt[\mathscr{r}]{\dfrac{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i} - \prod_{i=1}^{k}\left(1-\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i} + \prod_{i=1}^{k}(\varphi-1)\left(1-\left(\dfrac{\alpha_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}} \right) \right. \\
\left. \mathscr{b}\left(\dfrac{\prod_{i=1}^{k}\varphi\left(\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}{\sqrt[\mathscr{r}]{\prod_{i=1}^{k}\left(1+(\varphi-1)\left(\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_i} - \prod_{i=1}^{k}(\varphi-1)\left(1-\left(\dfrac{\beta_i}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_i}}} \right), \right)
\end{array} \right\}
$$

$$
\oplus_H \left\{ \begin{array}{l}
\left(\mathscr{b}\left(\dfrac{\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)\right)^{\Psi_{k+1}} - \left(1-\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)\right)^{\Psi_{k+1}}}{\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)\right)^{\Psi_{k+1}} + (\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)^{\Psi_{k+1}}} \right), \right. \\
\left. \mathscr{b}\left(\dfrac{\varphi\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)^{\Psi_{k+1}}}{\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)\right)^{\Psi_{k+1}} - (\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_{k+1})}}}{\mathscr{b}}\right)^{\Psi_{k+1}}} \right) \right), \\
\left(\mathscr{b}\left(\sqrt[\mathscr{r}]{\dfrac{\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_{k+1}} - \left(1-\left(\left(\dfrac{\alpha_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_{k+1}}}{\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_{k+1}} + (\varphi-1)\left(1-\left(\dfrac{\alpha_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_{k+1}}}} \right) \right. \\
\left. \mathscr{b}\left(\dfrac{\varphi\left(\left(\dfrac{\beta_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_{k+1}}}{\sqrt[\mathscr{r}]{\left(1+(\varphi-1)\left(\left(\dfrac{\beta_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)\right)^{\Psi_{k+1}} - (\varphi-1)\left(1-\left(\dfrac{\beta_{k+1}}{\mathscr{b}}\right)^{\mathscr{r}}\right)^{\Psi_{k+1}}}} \right), \right)
\end{array} \right\}
$$

$$= \left\{ \begin{array}{l} \left(\mathscr{b}\left(\frac{\prod_{i=1}^{k+1}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{k+1}\left(1-\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i}}{\prod_{i=1}^{k+1}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} + \prod_{i=1}^{k+1}(\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right), \; \mathscr{b}\left(\frac{\prod_{i=1}^{k}\varphi\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)^{\Psi_i}}{\prod_{i=1}^{k+1}\left(1+(\varphi-1)\left(\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)\right)^{\Psi_i} - \prod_{i=1}^{k+1}(\varphi-1)\left(1-\frac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_i)}}}{\mathscr{b}}\right)^{\Psi_i}} \right) \right), \\ \left(\mathscr{b}\left(\sqrt[r]{\frac{\prod_{i=1}^{k+1}\left(1+(\varphi-1)\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} - \prod_{i=1}^{k+1}\left(1-\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i}}{\prod_{i=1}^{k+1}\left(1+(\varphi-1)\left(\left(\frac{\alpha_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} + \prod_{i=1}^{k+1}(\varphi-1)\left(1-\left(\frac{\alpha_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}} \right) \right. \\ \left. \mathscr{b}\left(\frac{\prod_{i=1}^{k+1}\varphi\left(\left(\frac{\beta_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}{\sqrt[r]{\prod_{i=1}^{k+1}\left(1+(\varphi-1)\left(\left(\frac{\beta_i}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_i} - \prod_{i=1}^{k+1}(\varphi-1)\left(1-\left(\frac{\beta_i}{\mathscr{b}}\right)^{r}\right)^{\Psi_i}}} \right), \right) \end{array} \right\}$$

3. *TOPSIS Method under Non-Linear Diophantine Linguistic Information*

Making decisions is important in real life, and there are numerous ways to do it. In this study, we develop a unique model for non-linear Diophantine linguistic information based on the TOPSIS method. Let $\hbar = \{\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_n\}$ be the set of n alternatives and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \dots, \mathcal{C}_m\}$ be the set of m criteria. Let $\mathbb{E}_r | r = 1,2,3, \dots . l$ be the expert opinions. The TOPSIS method for non-linear Diophantine linguistic information contains of the resulting steps:

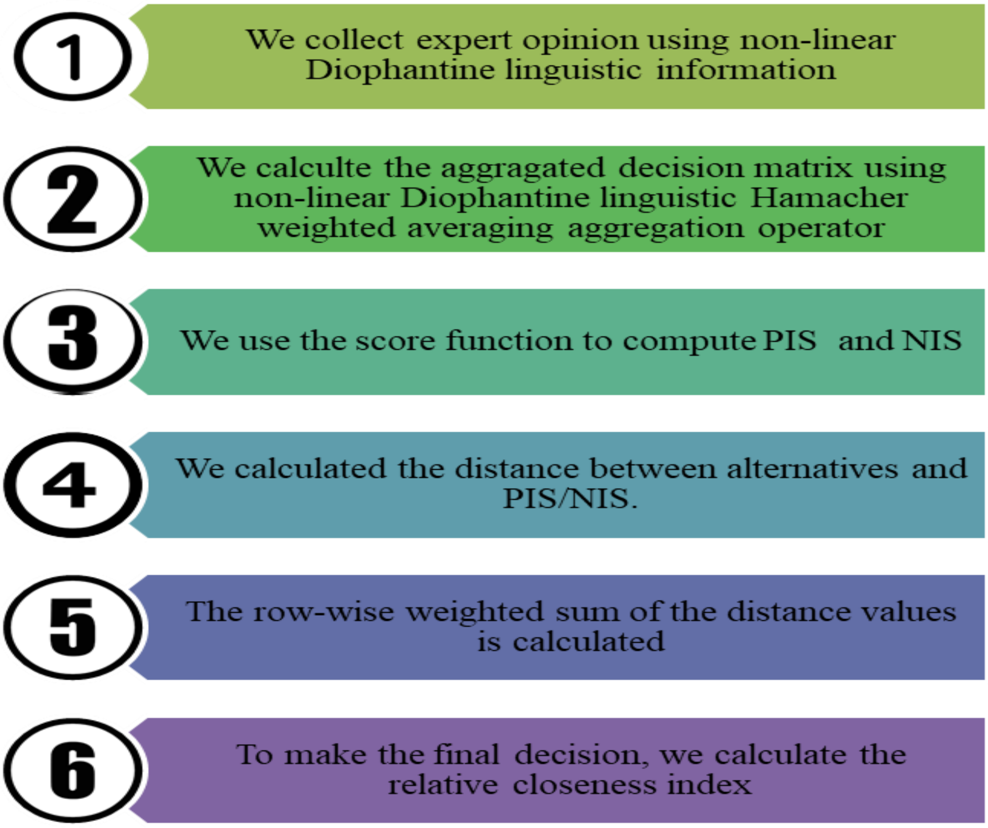

Figure 2: Flowchart of the proposed method

Step 1: We collect expert judgment using non-linear Diophantine linguistic information

$$\mathbb{E}_r = \begin{bmatrix} \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{11})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{11})}}\right), \\ (\alpha_{11}, \beta_{11})\end{matrix}\right\} & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{12})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{12})}}\right), \\ (\alpha_{12}, \beta_{12})\end{matrix}\right\} & \dots & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{1m})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{1m})}}\right), \\ (\alpha_{1m}, \beta_{1m})\end{matrix}\right\} \\ \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{21})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{21})}}\right), \\ (\alpha_{21}, \beta_{21})\end{matrix}\right\} & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{22})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{22})}}\right), \\ (\alpha_{22}, \beta_{22})\end{matrix}\right\} & \dots & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{2m})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{2m})}}\right), \\ (\alpha_{2m}, \beta_{2m})\end{matrix}\right\} \\ \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{31})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{31})}}\right), \\ (\alpha_{31}, \beta_{31})\end{matrix}\right\} & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{32})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{32})}}\right), \\ (\alpha_{32}, \beta_{32})\end{matrix}\right\} & \dots & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{3m})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{3m})}}\right), \\ (\alpha_{3m}, \beta_{3m})\end{matrix}\right\} \\ \vdots & \vdots & & \vdots \\ \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{n1})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{n1})}}\right), \\ (\alpha_{n1}, \beta_{n1})\end{matrix}\right\} & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{n2})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{n2})}}\right), \\ (\alpha_{n2}, \beta_{n2})\end{matrix}\right\} & \begin{matrix}\vdots \\ \dots\end{matrix} & \left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{nm})}}, \mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{nm})}}\right), \\ (\alpha_{nm}, \beta_{nm})\end{matrix}\right\} \end{bmatrix}$$

Step 2: We combine these experts$\mathbb{E}_{\mathscr{f}} | \mathscr{f} = 1,2,3, \dots . l$ opinion with known corresponding weights $\Psi = (\Psi_1, \Psi_2, \Psi_3, \dots, \Psi_l)^3$with condition that $\Psi_{\mathscr{f}} > 0$ and $\sum_{\mathscr{f}=1}^{l} \Psi_{\mathscr{f}} = 1$ by suing the N-LDLHWAA operator.

$$\text{N-LDLHWAA}(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, .., \mathfrak{J}_l) = \oplus_{\mathscr{f}=1}^{l} \Psi_{\mathscr{f}} \mathfrak{J}_{ij}$$

$$= \left\{ \begin{matrix} \left(\begin{matrix} \mathscr{b}\left(\dfrac{ \prod_{\mathscr{f}=1}^{l}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)\right)^{\Psi_{\mathscr{f}}} - \prod_{\mathscr{f}=1}^{l}\left(1-\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)\right)^{\Psi_{\mathscr{f}}} }{ \prod_{\mathscr{f}=1}^{l}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)\right)^{\Psi_{\mathscr{f}}} + \prod_{\mathscr{f}=1}^{l}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)^{\Psi_{\mathscr{f}}} } \right), \\ \mathscr{b}\left(\dfrac{ \prod_{\mathscr{f}=1}^{l}\varphi\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)^{\Psi_{\mathscr{f}}} }{ \prod_{\mathscr{f}=1}^{l}\left(1+(\varphi-1)\left(\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)\right)^{\Psi_{\mathscr{f}}} - \prod_{\mathscr{f}=1}^{l}(\varphi-1)\left(1-\dfrac{\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{ij})}}}{\mathscr{b}}\right)^{\Psi_{\mathscr{f}}} } \right) \end{matrix} \right), \\ \left(\begin{matrix} \mathscr{b}\left(\sqrt[r]{ \dfrac{ \prod_{\mathscr{f}=1}^{l}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_{ij}}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_{\mathscr{f}}} - \prod_{\mathscr{f}=1}^{l}\left(1-\left(\left(\dfrac{\alpha_{ij}}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_{\mathscr{f}}} }{ \prod_{\mathscr{f}=1}^{l}\left(1+(\varphi-1)\left(\left(\dfrac{\alpha_{ij}}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_{\mathscr{f}}} + \prod_{\mathscr{f}=1}^{l}(\varphi-1)\left(1-\left(\dfrac{\alpha_{ij}}{\mathscr{b}}\right)^{r}\right)^{\Psi_{\mathscr{f}}} } } \right) \\ \mathscr{b}\left(\dfrac{ \prod_{\mathscr{f}=1}^{l}\varphi\left(\left(\dfrac{\beta_{ij}}{\mathscr{b}}\right)^{r}\right)^{\Psi_{\mathscr{f}}} }{ \sqrt[r]{ \prod_{\mathscr{f}=1}^{l}\left(1+(\varphi-1)\left(\left(\dfrac{\beta_{ij}}{\mathscr{b}}\right)^{r}\right)\right)^{\Psi_{\mathscr{f}}} - \prod_{\mathscr{f}=1}^{l}(\varphi-1)\left(1-\left(\dfrac{\beta_{ij}}{\mathscr{b}}\right)^{r}\right)^{\Psi_{\mathscr{f}}} } } \right), \end{matrix} \right) \end{matrix} \right\}$$

Step 3: We use the score function to compute PIS $\left(\mathfrak{J}_j{}^{+}\right) = \left\{\begin{matrix}\left(\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_j)}}\right)^{+}, \left(\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_j)}}\right)^{+}\right), \\ \left((\alpha_j)^{+}, (\beta_j)^{+}\right)\end{matrix}\right\}$ and NIS $\left(\mathfrak{J}_j{}^{-}\right) = \left\{\begin{matrix}\left(\left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_j)}}\right)^{-}, \left(\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_i)}}\right)^{-}\right), \\ \left((\alpha_j)^{-}, (\beta_j)^{-}\right)\end{matrix}\right\}$ in order to process the model.

Step 4: Using the following formulas, we calculated the distance between alternatives and PIS/NIS.

$$\mathcal{T}_{ij}{}^{+} = \frac{1}{2}\left\{\begin{matrix} \left|\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_{ij})}} - \left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{J}_j)}}\right)^{+}\right| + \left|\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_{ij})}} - \left(\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{J}_j)}}\right)^{+}\right| \\ \left|\alpha_{ij} - (\alpha_j)^{+}\right| + \left|\beta_{ij} - (\beta_j)^{+}\right| \end{matrix}\right\}$$

$$\mathcal{T}_{ij}^{\ -} = \frac{1}{2}\left\{\begin{matrix}\left|\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_{ij})}} - \left(\mathcal{S}_{\mathfrak{m}_{\Omega(\mathfrak{I}_{j})}}\right)^{-}\right| + \left|\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_{ij})}} - \left(\mathcal{S}_{\mathfrak{n}_{\Omega(\mathfrak{I}_{j})}}\right)^{-}\right| \\ \left|\alpha_{ij} - (\alpha_j)^{-}\right| + \left|\beta_{ij} - (\beta_j)^{-}\right|\end{matrix}\right\}$$

Step 5: The following formals are used to compute the row-wise weighted totality of the distance values.

$$\overline{\overline{\dddot{w}}}_{\imath}^{\,+} = \sum_{j=1}^{m} \overline{\overline{\Psi}}_{j}\mathcal{T}_{ij}^{\ +}$$

$$\overline{\overline{\dddot{w}}}_{\imath}^{\,-} = \sum_{j=1}^{m} \overline{\overline{\Psi}}_{j}\mathcal{T}_{ij}^{\ -}$$

Step 6: To make the final decision, we calculate the relative closeness index using the following formulas.

$$\mathcal{h}_i = \frac{\overline{\overline{\dddot{w}}}_{\imath}^{\,-}}{\left(\overline{\overline{\dddot{w}}}_{\imath}^{\,+} + \overline{\overline{\dddot{w}}}_{\imath}^{\,-}\right)}$$

Figure 2 illustrates the proposed method's step-by-step processes, from expert information to final decision.

4. *Case study*

The selection of the best appropriate mathematical thinking tool [23-26] for problem solving is a key component of teaching methodology. Selecting the right tool will make solving the problem easier and more accurate.

We analyze the five different mathematical thinking tools based four key criteria:

Abstraction-($\mathcal{h}_1$): The process of generalizing mathematical ideas by eliminating particular specifics or real-world settings is known as abstraction in mathematical thought. It enables mathematicians to concentrate on the fundamental links and structure of mathematical concepts, enabling them to be applied to a wider variety of issues.

Generalization-($\mathcal{h}_2$): While verification guarantees the dependability of these models and generalizations, generalization is essential to mathematical reasoning because it converts particular facts into universal principles that direct model building, analysis, and theoretical breakthroughs.

Visualization-($\mathcal{h}_3$): Given their many uses in mathematics education, visualizations can improve students' comprehension of abstract ideas, their ability to draw conclusions, solve problems, build connections between mathematical ideas, and help them remember information.

Logical Reasoning-($\mathcal{h}_4$): It entails identifying trends, making connections, and using reasoning to solve problems. By decomposing complicated circumstances into smaller, easier-to-manage parts, this way of thinking helps kids comprehend and analyze them.

Modeling-($\mathcal{h}_5$): Mathematical modeling is the process of applying mathematics to depict and analyze real-world events in order to better understand, anticipate, or solve issues. It is reducing difficult circumstances into a mathematical framework, frequently using equations and variables, in order to acquire insights and maybe find answers.

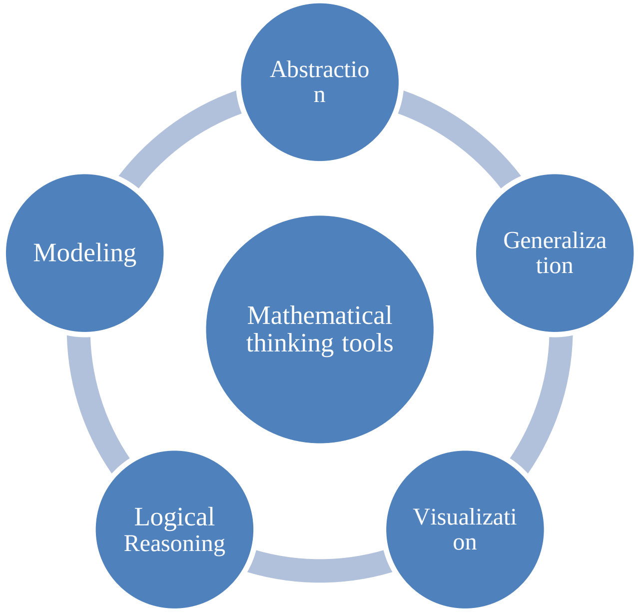

Figure 3: Five different mathematical thinking tools

Figure 3 show the five different mathematical thinking tools

- Participation of Students-($\mathcal{C}_1$)
- Theoretical Knowledge-($\mathcal{C}_2$)
- Implementation-($\mathcal{C}_3$)
- Transferability-($\mathcal{C}_4$)

5. Results Regarding Mathematical Thinking Tools

This section describes the proposed framework of non-linear Diophantine linguistic information for mathematical thinking tools.

Step 1: We non-linear Diophantine linguistic information to gather expert opinions on mathematical thinking tools. The findings are provided in Tables 1, 2, and 3.

Table 1: Expert 1 information

	$\boldsymbol{\mathcal{C}_1}$	$\boldsymbol{\mathcal{C}_2}$	$\boldsymbol{\mathcal{C}_3}$	$\boldsymbol{\mathcal{C}_4}$
$\boldsymbol{\hbar_1}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_2},\mathcal{S}_{\mathfrak{n}_4}\right),\\(4,5)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_3}\right),\\(4,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_2},\mathcal{S}_{\mathfrak{n}_1}\right),\\(5,7)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_7}\right),\\(1,3)\end{matrix}\right\}$
$\boldsymbol{\hbar_2}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_7},\mathcal{S}_{\mathfrak{n}_4}\right),\\(2,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_4}\right),\\(4,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_5}\right),\\(1,3)\end{matrix}\right\}$
$\boldsymbol{\hbar_3}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_5}\right),\\(2,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_2},\mathcal{S}_{\mathfrak{n}_1}\right),\\(1,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_1}\right),\\(2,7)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_1},\mathcal{S}_{\mathfrak{n}_1}\right),\\(5,5)\end{matrix}\right\}$
$\boldsymbol{\hbar_4}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_3}\right),\\(6,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_2}\right),\\(2,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_1}\right),\\(1,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_7},\mathcal{S}_{\mathfrak{n}_5}\right),\\(4,5)\end{matrix}\right\}$
$\boldsymbol{\hbar_5}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_7},\mathcal{S}_{\mathfrak{n}_1}\right),\\(4,4)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_5}\right),\\(5,7)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_6},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_6},\mathcal{S}_{\mathfrak{n}_1}\right),\\(1,5)\end{matrix}\right\}$

Table 2: Expert 2 information

	$\boldsymbol{\mathcal{C}_1}$	$\boldsymbol{\mathcal{C}_2}$	$\boldsymbol{\mathcal{C}_3}$	$\boldsymbol{\mathcal{C}_4}$
$\boldsymbol{\hbar_1}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_1},\mathcal{S}_{\mathfrak{n}_2}\right),\\(2,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_4}\right),\\(2,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_7}\right),\\(5,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_3}\right),\\(2,3)\end{matrix}\right\}$
$\boldsymbol{\hbar_2}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_6},\mathcal{S}_{\mathfrak{n}_5}\right),\\(3,6)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_1}\right),\\(2,4)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_5}\right),\\(4,2)\end{matrix}\right\}$
$\boldsymbol{\hbar_3}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,4)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_3}\right),\\(3,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_2}\right),\\(1,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_5}\right),\\(3,2)\end{matrix}\right\}$
$\boldsymbol{\hbar_4}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_3}\right),\\(4,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_1}\right),\\(3,4)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_7}\right),\\(6,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_2}\right),\\(2,3)\end{matrix}\right\}$
$\boldsymbol{\hbar_5}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_7}\right),\\(1,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_2}\right),\\(4\ 2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_4}\right),\\(5,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_4}\right),\\(5,3)\end{matrix}\right\}$

Table 3: Expert 3 information

	$\boldsymbol{\mathcal{C}_1}$	$\boldsymbol{\mathcal{C}_2}$	$\boldsymbol{\mathcal{C}_3}$	$\boldsymbol{\mathcal{C}_4}$
$\boldsymbol{\hbar_1}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_2},\mathcal{S}_{\mathfrak{n}_1}\right),\\(5,7)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_3}\right),\\(2,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_3}\right),\\(4,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_4}\right),\\(2,1)\end{matrix}\right\}$
$\boldsymbol{\hbar_2}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_5}\right),\\(4,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_4}\right),\\(4,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,2)\end{matrix}\right\}$
$\boldsymbol{\hbar_3}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_1}\right),\\(2,7)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_5}\right),\\(3,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_2},\mathcal{S}_{\mathfrak{n}_1}\right),\\(1,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_3}\right),\\(3,3)\end{matrix}\right\}$
$\boldsymbol{\hbar_4}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_1}\right),\\(1,2)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_2}\right),\\(2,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_2}\right),\\(2,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_1}\right),\\(3,4)\end{matrix}\right\}$
$\boldsymbol{\hbar_5}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_6},\mathcal{S}_{\mathfrak{n}_2}\right),\\(3,1)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_4},\mathcal{S}_{\mathfrak{n}_4}\right),\\(5,3)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_3},\mathcal{S}_{\mathfrak{n}_5}\right),\\(5,7)\end{matrix}\right\}$	$\left\{\begin{matrix}\left(\mathcal{S}_{\mathfrak{m}_5},\mathcal{S}_{\mathfrak{n}_2}\right),\\(4\ 2)\end{matrix}\right\}$

Step 2: We use the N-LDLHWAA operator to aggregate the experts' opinions with known corresponding weights (0.35, 0.35, and 0.30). The findings are provided in Table 4.

Table 4: Combined data

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\hbar_1$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.382}}, \mathcal{S}_{\mathfrak{n}_{2.799}}\right), (1.947, 7.241)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.047}}, \mathcal{S}_{\mathfrak{n}_{5.136}}\right), (1.901, 2.204)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.617}}, \mathcal{S}_{\mathfrak{n}_{5.873}}\right), (2.593, 6.607)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.281}}, \mathcal{S}_{\mathfrak{n}_{7.115}}\right), (1.287, 2.246)\right\}$
$\hbar_2$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{3.887}}, \mathcal{S}_{\mathfrak{n}_{6.042}}\right), (1.593, 4.523)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.344}}, \mathcal{S}_{\mathfrak{n}_{5.612}}\right), (2.056, 1.86)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.919}}, \mathcal{S}_{\mathfrak{n}_{2.550}}\right), (1.725, 3.405)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.904}}, \mathcal{S}_{\mathfrak{n}_{7.093}}\right), (1.485, 3.024)\right\}$
$\hbar_3$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.021}}, \mathcal{S}_{\mathfrak{n}_{3.307}}\right), (1.554, 3.99)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.814}}, \mathcal{S}_{\mathfrak{n}_{3.578}}\right), (1.403, 2.267)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.866}}, \mathcal{S}_{\mathfrak{n}_{1.451}}\right), (1.334, 5.197)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.522}}, \mathcal{S}_{\mathfrak{n}_{3.752}}\right), (2.338, 4.987)\right\}$
$\hbar_4$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.056}}, \mathcal{S}_{\mathfrak{n}_{2.877}}\right), (2.815, 1.86)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.923}}, \mathcal{S}_{\mathfrak{n}_{1.860}}\right), (1.554, 3.178)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.041}}, \mathcal{S}_{\mathfrak{n}_{4.993}}\right), (1.646, 2.363)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{3.510}}, \mathcal{S}_{\mathfrak{n}_{3.307}}\right), (1.935, 6.952)\right\}$
$\hbar_5$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{3.236}}, \mathcal{S}_{\mathfrak{n}_{4.993}}\right), (1.830, 3.405)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.038}}, \mathcal{S}_{\mathfrak{n}_{5.743}}\right), (2.432, 7.761)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.743}}, \mathcal{S}_{\mathfrak{n}_{5.612}}\right), (2.028, 4.353)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.805}}, \mathcal{S}_{\mathfrak{n}_{2.686}}\right), (1.605,5.126)\right\}$

Step 3: To process the model, we use the score function to compute the PIS and NIS values. The findings are presented in Table 5.

Table 5: PIS and NIS values

	PIS	NIS
$\mathcal{C}_1$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.382}}, \mathcal{S}_{\mathfrak{n}_{2.799}}\right), (1.947, 7.241)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.056}}, \mathcal{S}_{\mathfrak{n}_{2.877}}\right), (2.815, 1.86)\right\}$
$\mathcal{C}_2$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.038}}, \mathcal{S}_{\mathfrak{n}_{5.743}}\right), (2.432, 7.761)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.923}}, \mathcal{S}_{\mathfrak{n}_{1.860}}\right), (1.554, 3.178)\right\}$
$\mathcal{C}_3$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.617}}, \mathcal{S}_{\mathfrak{n}_{5.873}}\right), (2.593, 6.607)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.919}}, \mathcal{S}_{\mathfrak{n}_{2.550}}\right), (1.725, 3.405)\right\}$
$\mathcal{C}_4$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{1.904}}, \mathcal{S}_{\mathfrak{n}_{7.093}}\right), (1.485, 3.024)\right\}$	$\left\{\left(\mathcal{S}_{\mathfrak{m}_{2.805}}, \mathcal{S}_{\mathfrak{n}_{2.686}}\right), (1.605,5.126)\right\}$

Step 4: In this stage, we determined the distance between alternatives and PIS/NIS in order to perform the model.
Step 5: In this stage, the row-wise weighted sum of distance values is determined.
Step 6: To make the final decision, we calculate the relative closeness index, and the decision will be made based on it. The findings are provided in Table 6.

Table 6: Relative closeness index

	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	$\hbar_5$
$\overline{\overline{\dddot{w}_{\iota}}}^{+}$	0.1330	0.3472	0.4321	0.5190	0.3097
$\overline{\overline{\dddot{w}_{\iota}}}^{-}$	0.4302	0.3624	0.2070	0.1116	0.3117
$\hbar_i$	0.7639	0.5107	0.3240	0.1770	0.5017

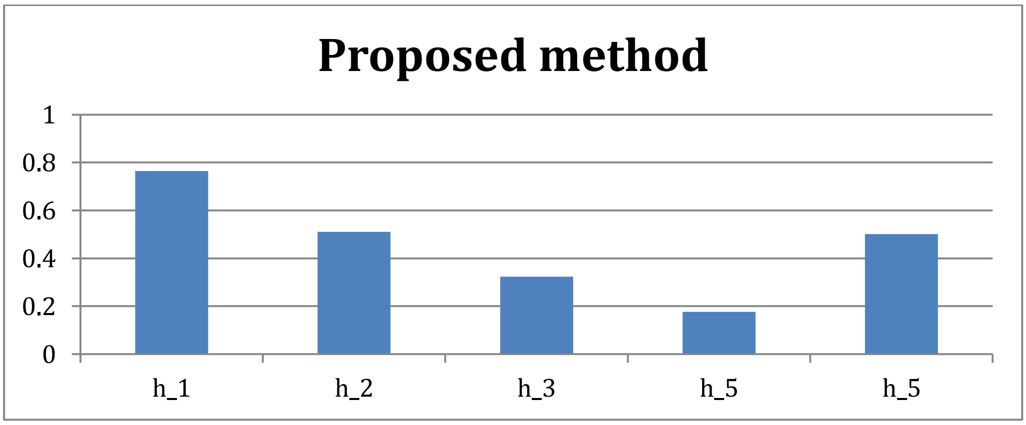

Figure 4: The graphical representation of the proposed framework's outcomes

The proposed framework demonstrates that Abstraction is the greatest mathematical thinking tool based on expert opinion, as shown in Table 6 and Figure 4.

6. *Comparative Analysis*

This section discusses the comparison between the proposed method and existing MCDM approaches. To assess the proposed model's accuracy and performance, we compared it with existing MCDM approaches such as GRA [27], EDAS [28], and WASPAS [29], using the same aggregated data and criterion weights (0.24, 0.26, 0.22, and 0.28). Use the above methods to compute the final data, and the results are shown in Table 7.

Table 7: The results based on existing MCDM methods

Methods	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	$\hbar_5$
GRA	0.6395	0.4962	0.3938	0.3099	0.4913
EDAS	0.9384	0.4761	0.2711	0.1564	0.8064
WASPAS	0.3618	0.2717	0.2395	0.1845	0.3187

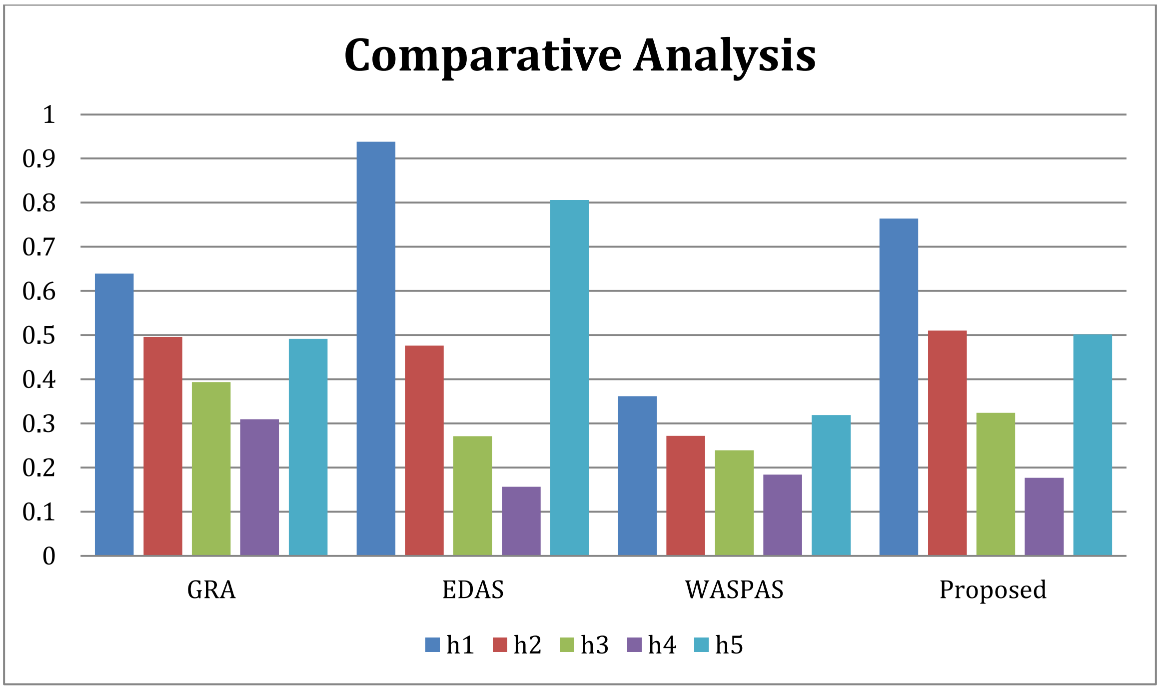

Figure 5: Graphical representation of the comparative analysis

Based on expert judgment, the comparative analysis shows that Abstraction is the best mathematical thinking tool, as seen in Figure 5 and Table 8. Comparative analysis demonstrates that the proposed framework is accurate and adaptable to real-life decision-making difficulties.

Table 8: The ranking results

Methods	Ranking	Best
GRA	$\hbar_1 > \hbar_2 > \hbar_5 > \hbar_3 > \hbar_4$	Abstraction-($\hbar_1$)
EDAS	$\hbar_1 > \hbar_5 > \hbar_2 > \hbar_3 > \hbar_4$	Abstraction-($\hbar_1$)
WASPAS	$\hbar_1 > \hbar_5 > \hbar_2 > \hbar_3 > \hbar_4$	Abstraction-($\hbar_1$)
Proposed	$\hbar_1 > \hbar_2 > \hbar_5 > \hbar_3 > \hbar_4$	Abstraction-($\hbar_1$)

7. *Conclusion*

Non-linear Diophantine linguistic term sets are a novel concept that we developed in this work. They are a combination of linguistic term sets and non-linear Diophantine fuzzy sets. We started by going over the basic concept of fuzzy sets and how it may be generalized. The non-linear Diophantine linguistic term sets, along with their score and accuracy functions, were then defined after discussing both linear and nonlinear Diophantine fuzzy sets. Now, we applied Hamacher t-norm and t-conorm to construct the aggregation operator along with the operational rules for the proposed concept. Also, we applied Hamacher aggregation operator in the modification of TOPSIS method for non-linear Diophantine linguistic term sets. Then, we applied this approach to in making decisions on mathematical thinking tools and gathered reliable opinions from three experts. We applied the modified TOPSIS to the final data after the data from these experts was aggregated using Hamacher aggregation operator. Abstraction emerges as the most optimal mathematical tool within the given framework. Finally, we tested the accuracy and applicability of the model that we proposed by comparing it to other existing approaches. The comparative analysis provided demonstrates the accuracy and adaptability of the proposed framework to practical decision-making scenarios.

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Human Participants and/or Animals*:* This research did not involve any human participants or animals.

Ethics approval: This paper does not contain any studies with animals performed by any of the authors.

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